Complex Analysis Notes for ET4-3

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1 Singularities of rational functions

Consider two functions f and g both defined on a domain $D \subset \mathbb{C}$, and analytic on D. Define $h(z) = \frac{f(z)}{g(z)}$ in all points of D where $g \neq 0$.

We say that $z_0 \in D$ is a zero of order $k \geq 0$ for f if

$$f(z_0) = f'(z_0) = \cdots = f^{(k-1)}(z_0) = 0 \quad \text{and} \quad f^{(k)}(z_0) \neq 0. \tag{1.1}$$

With the same definition, a point $z_0 \in D$ is a zero of order $m \geq 0$ for g if

$$g(z_0) = g'(z_0) = \cdots = g^{(k-1)}(z_0) = 0 \quad \text{and} \quad g^{(k)}(z_0) \neq 0. \tag{1.2}$$

Note that if for example $f(z_0) \neq 0$, then we either say that z_0 is not a zero, or that z_0 is a zeroth order zero. In this way, we can classify all points of D.

Now choose an arbitrary point $z_0 \in D$, and assume that it is a zero of order k for f. Since f is analytic in z_0, we can expand f in a Taylor series in a small disk around z_0:

$$\begin{aligned} f(z) &= \sum_{n\geq 0} \frac{1}{n!}\frac{df^{(n)}}{dz^n}(z_0)\,(z-z_0)^n = \sum_{n\geq k} \frac{1}{n!}\frac{df^{(n)}}{dz^n}(z_0)\,(z-z_0)^n \\ &= (z-z_0)^k \sum_{n\geq k} \frac{1}{n!}\frac{df^{(n)}}{dz^n}(z_0)\,(z-z_0)^{n-k} \\ &= (z-z_0)^k \sum_{n\geq 0} \frac{1}{(n+k)!}\frac{df^{(n+k)}}{dz^{n+k}}(z_0)\,(z-z_0)^n. \end{aligned} \tag{1.3}$$

Thus we may write:

$$\begin{aligned} &\tilde{f}(z) := \sum_{n\geq 0} \frac{1}{(n+k)!}\frac{df^{(n+k)}}{dz^{n+k}}(z_0)\,(z-z_0)^n, \quad \frac{d\tilde{f}^{(n)}}{dz^n}(z_0) = \frac{n!}{(n+k)!}\frac{df^{(n+k)}}{dz^{n+k}}(z_0) \\ &f(z) = (z-z_0)^k \tilde{f}(z). \end{aligned} \tag{1.4}$$

Now let us assume that z_0 is a zero of order m for g. Reasoning as above, we may write:

$$\begin{aligned} &\tilde{g}(z) := \sum_{n\geq 0} \frac{1}{(n+m)!}\frac{dg^{(n+m)}}{dz^{n+m}}(z_0)\,(z-z_0)^n, \quad \frac{d\tilde{g}^{(n)}}{dz^n}(z_0) = \frac{n!}{(n+m)!}\frac{dg^{(n+m)}}{dz^{n+m}}(z_0) \\ &g(z) = (z-z_0)^m \tilde{g}(z). \end{aligned} \tag{1.5}$$

Therefore, the function h can be expressed in a neighborhood of z_0 as:

$$h(z) = (z-z_0)^{k-m}\tilde{h}(z), \quad \tilde{h}(z) := \frac{\tilde{f}(z)}{\tilde{g}(z)}, \quad \tilde{h}(z_0) = \frac{\tilde{f}(z_0)}{\tilde{g}(z_0)} = \frac{m!}{k!}\,\frac{\frac{df^{(k)}}{dz^k}(z_0)}{\frac{dg^{(m)}}{dz^m}(z_0)} \neq 0. \tag{1.6}$$

At this moment we can fully investigate the nature of the point z_0. There are two distinct situations:

1. If $k \geq m$, then z_0 is a zero of order $k-m$ for h; indeed, if we differentiate the product $(z-z_0)^{k-m}\tilde{h}(z)$ less than $k-m$ times, then all terms we get will contain a positive power of $(z-z_0)$. Then if we put $z = z_0$ they will all be zero. The first time when this is no longer true, is when we differentiate precisely $k-m$ times, and all derivatives fall on the factor $(z-z_0)^{k-m}$. In fact, one can prove that

$$\frac{dh^{(k-m)}}{dz^{k-m}}(z_0) = (k-m)!\,\tilde{h}(z_0) \neq 0,$$

which finishes the proof that z_0 is a zero of order $k-m$ for h.

2. If $k < m$, then z_0 is a pole of order $m-k$ for h; indeed, according to the definition, we have:

$$\lim_{z\to z_0}(z-z_0)^{m-k}h(z) = \tilde{h}(z_0) \neq 0.$$

2 Laurent series for rational functions

Assume that we are in the situation in which $k < m$, or in other words $m-k > 0$. Since $\tilde{h}$ is an analytic function around z_0, we can expand it in a Taylor series given by:

$$\begin{aligned}\tilde{h}(z) &= \sum_{n\geq 0}\frac{1}{n!}\frac{d\tilde{h}^{(n)}}{dz^n}(z_0)\,(z-z_0)^n \\ &= \sum_{n=0}^{m-k-1}\frac{1}{n!}\frac{d\tilde{h}^{(n)}}{dz^n}(z_0)\,(z-z_0)^n + \sum_{n\geq m-k}\frac{1}{n!}\frac{d\tilde{h}^{(n)}}{dz^n}(z_0)\,(z-z_0)^n \\ &= \sum_{n=0}^{m-k-1}\frac{1}{n!}\frac{d\tilde{h}^{(n)}}{dz^n}(z_0)\,(z-z_0)^n + (z-z_0)^{m-k}\sum_{n\geq 0}\frac{1}{(n+m-k)!}\frac{d\tilde{h}^{(n+m-k)}}{dz^{n+m-k}}(z_0)\,(z-z_0)^n.\end{aligned} \tag{2.7}$$

Thus:

$$\begin{aligned}h(z) &= \frac{1}{(z-z_0)^{m-k}}\tilde{h}(z) \\ &= \sum_{n=0}^{m-k-1}\frac{1}{n!}\frac{d\tilde{h}^{(n)}}{dz^n}(z_0)\,\frac{1}{(z-z_0)^{m-k-n}} + \sum_{n\geq 0}\frac{1}{(n+m-k)!}\frac{d\tilde{h}^{(n+m-k)}}{dz^{n+m-k}}(z_0)\,(z-z_0)^n \\ &= \frac{\tilde{h}(z_0)}{(z-z_0)^{m-k}} + \cdots + \frac{\frac{1}{(m-k-1)!}\frac{d\tilde{h}^{(m-k-1)}}{dz^{m-k-1}}(z_0)}{z-z_0} \\ &\quad + \sum_{n\geq 0}\frac{1}{(n+m-k)!}\frac{d\tilde{h}^{(n+m-k)}}{dz^{n+m-k}}(z_0)\,(z-z_0)^n.\end{aligned} \tag{2.8}$$

We know that the Laurent series of h around z_0 must be of the form:

$$h(z) = \sum_{n\geq 1}\frac{b_n}{(z-z_0)^n} + \sum_{n\geq 0}a_n(z-z_0)^n. \tag{2.9}$$

If we compare (2.9) with (2.8) we conclude the following:

$$b_n = 0 \quad \text{if} \quad n > m-k,$$

$$b_{m-k} = \tilde{h}(z_0) \neq 0, \qquad b_1 = \frac{1}{(m-k-1)!}\frac{d\tilde{h}^{(m-k-1)}}{dz^{m-k-1}}(z_0), \tag{2.10}$$

and finally

$$a_n = \frac{1}{(n+m-k)!}\frac{d\tilde{h}^{(n+m-k)}}{dz^{n+m-k}}(z_0), \quad n \geq 0.$$

3 Residue calculus

Let us compute the above coefficient b_1 when $m = k+1$; then (2.10) gives:

$$b_1 = \tilde{h}(z_0) = \frac{m!}{k!}\frac{\frac{df^{(k)}}{dz^k}(z_0)}{\frac{dg^{(m)}}{dz^m}(z_0)} = (k+1)\frac{\frac{df^{(k)}}{dz^k}(z_0)}{\frac{dg^{(k+1)}}{dz^{k+1}}(z_0)}. \tag{3.11}$$

But in general it is complicated to write down a formula for b_1. In different application it is easier to repeat the whole algorithm from the beginning than to follow some pre-determined formulas. Let us solve an exercise.

Exercise 3.1. *Consider the function*

$$h(z) = \frac{\{\sin(z)\}^2}{e^{z^4} - 1},$$

defined at all points where $e^{z^4} \neq 1$*. Show that* $z_0 = 0$ *is a pole of order* 2*, and compute the residue* b_1*.*

Solution. We have $f(z) = \{\sin(z)\}^2$ and $g(z) = e^{z^4} - 1$. Both functions are analytic on the whole complex plane.

Now let us see what is k. We have $f(0) = 0$, hence we must investigate $f'(0)$. We compute:

$$f'(z) = 2\sin(z)\cos(z) = \sin(2z), \qquad f'(0) = 0.$$

Then we continue with $f''(z) = 2\cos(2z)$, hence $f''(0) = 2 \neq 0$. Thus $k = 2$ and we may write

$$f(z) = z^2 \tilde{f}(z), \quad \frac{d\tilde{f}^{(n)}}{dz^n}(0) = \frac{n!}{(n+2)!}\frac{df^{(n+2)}}{dz^{n+2}}(0), \quad n \geq 0.$$

Let us find m. Using the formula $e^w = \sum_{n\geq 0} \frac{w^n}{n!}$, we have

$$g(z) = \sum_{n\geq 1} \frac{z^{4n}}{n!} = z^4 + \frac{z^8}{2} + \dots.$$

This means that $g(0) = g'(0) = g''(0) = g^{(3)}(0) = 0$ and $g^{(4)}(0) = 4! = 24 \neq 0$. Thus $m = 4$, and

$$g(z) = z^4 \tilde{g}(z), \quad \frac{d\tilde{g}^{(n)}}{dz^n}(0) = \frac{n!}{(n+4)!}\frac{dg^{(n+4)}}{dz^{n+4}}(0), \quad n \geq 0.$$

Now we may write:

$$h(z) = \frac{1}{z^2}\tilde{h}(z), \quad \tilde{h}(z) = \frac{\tilde{f}(z)}{\tilde{g}(z)}.$$

According to (2.10), we have $b_1 = \tilde{h}'(0)$. We have:

$$\tilde{h}'(0) = \frac{\tilde{f}'(0)\tilde{g}(0) - \tilde{f}(0)\tilde{g}'(0)}{\{\tilde{g}(0)\}^2}$$

in which we can insert $\tilde{g}(0) = \frac{1}{4!}\frac{dg^{(4)}}{dz^4}(0) = 1$, $\tilde{g}'(0) = \frac{1}{5!}\frac{dg^{(5)}}{dz^5}(0) = 0$, $\tilde{f}(0) = \frac{1}{2!}\frac{df^{(2)}}{dz^2}(0) = 1$ and $\tilde{f}'(0) = \frac{1}{3!}\frac{df^{(3)}}{dz^3}(0) = 0$. This gives $b_1 = 0$.

4 Some typical exam exercises

Exercise 4.1. *Find all the complex solutions of the equation* $e^{z^3} = 1$.

Solution. We know that the exponential function is $2\pi i$ periodic, thus z^3 must be of the form $2\pi i N$ with $N \in \mathbb{Z}$. There are three possibilities for N:

1. If $N = 0$, then the only solution is $z = 0$;

2. For each $N > 0$, let us solve the equation $z^3 = 2\pi i N = 2\pi N e^{i\pi/2}$. For each N we find three solutions:

$$z_k = (2\pi N)^{1/3} e^{i(\pi/6 + 2\pi k/3)}, \quad k \in \{0, 1, 2\}.$$

3. For each $N < 0$, let us solve the equation $z^3 = -2\pi i|N| = 2\pi |N| e^{-i\pi/2}$. This gives other three solutions:

$$z_k = (2\pi |N|)^{1/3} e^{i(-\pi/6 + 2\pi k/3)}, \quad k \in \{0, 1, 2\}.$$

Exercise 4.2. *Let* $f(z) = |z|^2 + \overline{z}$, *where* $z = x + iy$.

1. Find two real functions u *and* v *such that* $f(z) = u(x, y) + iv(x, y)$ *for all* z.

2. Is f *analytic?*

Solution.

1. We have $\overline{z} = x - iy$ and $|z|^2 = x^2 + y^2$, thus $u(x, y) = x + x^2 + y^2$ and $v(x, y) = -y$.

2. The function is not analytic, because the Cauchy-Riemann equations are not satisfied. For example, $\partial_x u = 1 + 2x$ is not identically equal with $\partial_y v = -1$.

Exercise 4.3. *Let* $f(z) = \overline{z}$, *where* $z = x + iy$. *Let* γ *be a circle of radius* 1, *centred at* $z_0 = 1 + i$, *and oriented anti-clockwise. Show that the path integral*

$$\int_\gamma f(z)dz = 2\pi i.$$

Is this result in contradiction with Cauchy's integral theorem?

Solution. We can parameterize the circle as $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + e^{it}$. Here $\gamma'(t) = ie^{it}$ and $z_0 = \sqrt{2}e^{i\pi/4}$. Then we have

$$\begin{aligned} \int_\gamma f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt = i\int_0^{2\pi} \overline{\sqrt{2}e^{i\pi/4} + e^{it}}\, e^{it}dt \\ &= i\int_0^{2\pi} (\sqrt{2}e^{-i\pi/4}e^{it} + 1)dt = 2\pi i. \end{aligned} \tag{4.12}$$

Thus the integral of f on a closed path is not zero. This is possible because f is not analytic, thus Cauchy's theorem is not contradicted.

Exercise 4.4. *Find the convergence radius of the power series* $\sum_{n\geq 0} \frac{2n^3+1}{n+1} z^n$.

Solution.

Let $a_n = \frac{2n^3+1}{n+1}$. This implies that $a_{n+1} = \frac{2(n+1)^3+1}{n+2}$. Then we have:

$$\frac{a_n}{a_{n+1}} = \frac{(2n^3 + 1)(n + 2)}{(2(n + 1)^3 + 1)(n + 1)} = \frac{(2 + 1/n^3)(1 + 2/n)}{(2(1 + 1/n)^3 + 1)(1 + 1/n)}, \quad n \geq 1.$$

Thus the radius of convergence is:

$$R = \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = 1.$$

Exercise 4.5. *Let the function $h(z) = \frac{\sin(z\pi)}{z^4-1}$ initially defined on all points where $z^4 \neq 1$. Show that h is analytic at $z = \pm 1$, and has two first order poles at $\pm i$.*

Solution. Let $f(z) = \sin(z\pi)$ and $g(z) = z^4 - 1$. The possible singularities of h are the solutions of the equation $z^4 = 1$. These points are given by

$$e^{k\pi i/2}, \quad k \in \{0, 1, 2, 3\},$$

or $z_1 = 1$, $z_2 = i$, $z_3 = -1$, $z_4 = -i$.

1. We show that the Laurent series of h near $z_1 = 1$ has all its b coefficients equal to zero. We have $f(1) = 0$, $f'(z) = \pi \cos(z\pi)$ thus $f'(1) = -\pi \neq 0$. Hence 1 is a first order zero for f and we can write $f(z) = (z-1)\tilde{f}(z)$, with $\tilde{f}(1) = f'(1) = -\pi$.

In a similar way, $g(1) = 0$, $g'(z) = 4z^3$ and $g'(1) = 4 \neq 0$, thus $g(z) = (z-1)\tilde{g}(z)$ with $\tilde{g}(1) = g'(1) = 4$. It means that $h(z) = \frac{\tilde{f}(z)}{\tilde{g}(z)}$, which is analytic around 1.

2. We can redo the same type of argument near $z_3 = -1$. We obtain $f(-1) = 0$, $f'(-1) = -\pi \neq 0$, $g(-1) = 0$ and $g'(-1) = -4 \neq 0$. Since -1 is a first order zero for both f and g, the function h is analytic near -1.

3. Let us investigate the Laurent series of h near $z_2 = i$. We have $f(i) = \sin(i\pi) \neq 0$, thus $\tilde{f}(z) = f(z)$. But $g(i) = 0$ and $g'(i) = -4i \neq 0$, which means $g(z) = (z-i)\tilde{g}(z)$ with $\tilde{g}(i) = -4i$. Here we are in the situation $k = 0$ and $m = 1$, see (2.8)-(2.10). Thus $h(z) = \frac{1}{z-i}\tilde{h}(z)$ and $b_1 = i\sin(i\pi)/4 \neq 0$.

4. The treatment of z_4 is similar with that of z_3.

Exercise 4.6. *Compute the integral*

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+9} dx.$$

Solution. Let us consider the function $h(z) = \frac{e^{iz}}{z^2+9}$. We have the equality:

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2+9} dx = \operatorname{Re}\left(\int_{-\infty}^{\infty} h(x)dx\right).$$

1. Let us show that he function h has two simple poles: one at $z_1 = 3i$ and the other one at $z_2 = -3i$. We have $f(z) = e^{iz}$ and $g(z) = z^2 + 9$. The only two points where g is zero, are z_1 and z_2. The function f has no zeroes. Since $g'(z) = 2z$, it means that z_1 and z_2 are first order zeroes. Near z_1 we can write $g(z) = (z-3i)\tilde{g}(z)$ with $\tilde{g}(z) = z + 3i$. It means that near $z_1 = 3i$ we can write:

$$h(z) = \frac{1}{z-3i}\,\frac{e^{iz}}{z+3i}, \quad \operatorname{Res}_{z=z_1}(h) = b_1 = \frac{e^{-3}}{6i}.$$

In a similar way, near $z_2 = -3i$ we have:

$$h(z) = \frac{1}{z+3i}\,\frac{e^{iz}}{z-3i}, \quad \operatorname{Res}_{z=z_2}(h) = b_1 = -\frac{e^{3}}{6i}.$$

In order to compute the integral of h we close the contour through the upper half plane and apply the residue theorem:

$$\int_{-\infty}^{\infty} h(x)dx = 2\pi i \operatorname{Res}_{z=z_1}(h) = \frac{\pi e^{-3}}{3}.$$

Exercise 4.7. *Compute the integral*

$$\int_0^{2\pi} \frac{\sin^2(\theta)}{5-4\cos(\theta)} d\theta.$$

Solution. We reason as in section 16.4, formula 2 on page 718. We have:

$$\cos(\theta) = \frac{1}{2}(z + 1/z), \quad \sin(\theta) = \frac{1}{2i}(z - 1/z), \quad d\theta = \frac{dz}{iz}.$$

Thus the integral can be rewritten as a path integral over the unit circle of the function

$$h(z) = \frac{-\frac{1}{4}(z - 1/z)^2}{5 - 2(z + 1/z)}\ \frac{1}{iz} = \quad \frac{(z^2 - 1)^2}{4iz^2(2z^2 - 5z + 2)}.$$

We need to identify the eventual singularities of h inside the unit circle. We have $f(z) = (z^2 - 1)^2$ and $g(z) = 4iz^2(2z^2 - 5z + 2)$. We can factorize $2z^2 - 5z + 2 = 2(z - 1/2)(z - 2)$. Thus g has two zeroes inside the unit circle: $z_1 = 0$ and $z_2 = 1/2$.

Let us show that $z_1 = 0$ is a pole of order two. Since $f(0) = 1 \neq 0$, we have $k = 0$. Moreover, $m = 2$ because we have $g(z) = z^2\tilde{g}(z)$ with $\tilde{g}(z) = 4i(2z^2 - 5z + 2)$ and $\tilde{g}(0) = 8i \neq 0$. Thus

$$h(z) = \frac{1}{z^2}\,\tilde{h}(z), \quad \tilde{h}(z) = \frac{(z^2 - 1)^2}{4i(2z^2 - 5z + 2)}.$$

Then according to (2.10) we have:

$$\mathrm{Res}_{z=0}h = b_1 = \tilde{h}'(0) = -\frac{5i}{16}.$$

Now let us treat $z_2 = 1/2$. Again, $f(1/2) = 9/16 \neq 0$, thus $k = 0$. Moreover, $m = 1$ because we have $g(z) = (z - 1/2)\tilde{g}(z)$ with $\tilde{g}(z) = 8iz^2(z - 2)$ and $\tilde{g}(1/2) = -3i \neq 0$. Thus

$$h(z) = \frac{1}{z - 1/2}\,\tilde{h}(z), \quad \tilde{h}(z) = \frac{(z^2 - 1)^2}{8iz^2(z - 2)}.$$

Then according to (2.10) we have:

$$\mathrm{Res}_{z=1/2}h = b_1 = \tilde{h}(1/2) = \frac{3i}{16}.$$

Then according to the residue theorem on page 715 we have:

$$\int_{|z|=1} h(z)dz = 2\pi i(\mathrm{Res}_{z=0}h + \mathrm{Res}_{z=1/2}h) = \frac{\pi}{4}.$$