Complex Analysis Notes

Horia Cornean, d.29/03/2011.

1 Some typical exam exercises

Exercise 1.1. *Find all complex solutions to the equation* $e^{z^2} = 1$.

Solution. We know that the exponential function is $2\pi i$ periodic, thus z^2 must be of the form $2\pi i N$ with $N \in \mathbb{Z}$. There are three possibilities for N:
1. If $N = 0$, then the only solution is $z = 0$;
2. For each $N > 0$, let us solve the equation $z^2 = 2\pi i N = 2\pi N e^{i\pi/2}$. For each positive N we find two solutions:

$$z_k = (2\pi N)^{1/2} e^{i(\pi/4+\pi k)} = (-1)^k (2\pi N)^{1/2} e^{i\pi/4} = (-1)^k (\pi N)^{1/2}(1+i), \quad k \in \{0, 1\}.$$

3. For each $N < 0$, let us solve the equation $z^2 = -2\pi i|N| = 2\pi|N|e^{-i\pi/2}$. This gives other two solutions:

$$z_k = (2\pi|N|)^{1/2} e^{i(-\pi/4+\pi k)} = (-1)^k (2\pi|N|)^{1/2} e^{-i\pi/4} = (-1)^k (\pi N)^{1/2}(1-i), \quad k \in \{0, 1, 2\}.$$

Exercise 1.2. *Let* $f(z) = |z|^2 + \overline{z}$, *where* $z = x + iy$.
1. Find two real functions u *and* v *such that* $f(z) = u(x, y) + iv(x, y)$ *for all* z.
2. Is f *analytic?*

Solution.
1. We have $\overline{z} = x - iy$ and $|z|^2 = x^2 + y^2$, thus $u(x, y) = x + x^2 + y^2$ and $v(x, y) = -y$.
2. The function is not analytic, because the Cauchy-Riemann equations are not satisfied. For example, $\partial_x u = 1 + 2x$ is not identically equal with $\partial_y v = -1$.

Exercise 1.3. *Let* $f(z) = \overline{z}$, *where* $z = x + iy$. *Let* γ *be a circle of radius* 1, *centred at* $z_0 = 1 + i$, *and oriented anti-clockwise. Show that the path integral*

$$\int_\gamma f(z)dz = 2\pi i.$$

Is this result in contradiction with Cauchy's integral theorem?

Solution. We can parameterize the circle as $\gamma : [0, 2\pi] \to \mathbb{C}$, $\gamma(t) = z_0 + e^{it}$. Here $\gamma'(t) = ie^{it}$ and $z_0 = \sqrt{2}e^{i\pi/4}$. Then we have

$$\begin{aligned}\int_\gamma f(z)dz &= \int_0^{2\pi} f(\gamma(t))\gamma'(t)dt = i\int_0^{2\pi} \overline{\sqrt{2}e^{i\pi/4} + e^{it}}\, e^{it}dt \\ &= i\int_0^{2\pi} (\sqrt{2}e^{-i\pi/4}e^{it} + 1)dt = 2\pi i. \qquad (1.1)\end{aligned}$$

Thus the integral of f on a closed path is not zero. This is possible because f is not analytic, thus Cauchy's theorem is not contradicted.

Exercise 1.4. *Find the convergence radius of the power series* $\sum_{n\geq 0}(1+i)^n \frac{2n^3+i}{(n+i)^2} z^n$.

Solution.
Let $a_n = (1+i)^n \frac{2n^3+i}{(n+i)^2}$. This implies that $a_{n+1} = (1+i)^{n+1} \frac{2(n+1)^3+i}{(n+1+i)^2}$. Then we have:

$$\frac{a_n}{a_{n+1}} = \frac{1}{1+i} \frac{(2n^3+i)(n+1+i)^2}{(2(n+1)^3+i)(n+i)^2} = \frac{1}{1+i} \frac{(2+i/n^3)(1+(1+i)/n)^2}{(2(1+1/n)^3+i/n^3)(1+i/n)^2}, \quad n \geq 1.$$

Thus the radius of convergence is:

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{\sqrt{2}}.$$

Exercise 1.5. *We use the notation from Theorem 1 on page 754 (Section 18.2 in Kreyszig). Assume that* D^* *is the image of the rectangle*

$$D = \left\{ [x,y] \in \mathbb{R}^2 : \ 0 \leq x \leq \frac{\pi}{2}, \ 0 \leq y \leq 1 \right\}$$

under the function $w = f(z) = \sin(z)$. *Assume that* $\Phi^*(u,v) = u^2 - v^2$. *Find the corresponding harmonic potential* Φ *in* D *and its boundary values.*

Solution.
We have $f(z) = w = u + iv = \frac{1}{2i}(e^{iz} - e^{-iz}) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$. Thus $\Phi(x,y) = \Phi^*(u(x,y), v(x,y)) = \sin^2(x)\cosh^2(y) - \cos^2(x)\sinh^2(y)$.

On the side of D defined by $0 \leq x \leq \pi/2$ and $y = 0$ we have the boundary value $\phi_1(x) = \sin^2(x)$. On the side defined by $0 \leq x \leq \pi/2$ and $y = 1$ we have $\phi_2(x) = \sin^2(x)\cosh^2(1) - \cos^2(x)\sinh^2(1)$. On the side defined by $x = 0$ and $0 \leq y \leq 1$ we have $\phi_3(y) = -\sinh^2(y)$. On the side given by $x = \pi/2$ and $0 \leq y \leq 1$ we have $\phi_4(y) = \cosh^2(y)$.