Path and surface integrals

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1 Path integrals

1.1 General things about paths

A three dimensional path, or curve, is parametrized by just one real variable. Mathematically, a path is the range of a function

$$[a,b] \ni u \mapsto \vec{r}(u) \in \mathbb{R}^3.$$

Some important examples:

- A straight line starting at point $\vec{r}_A := (x_A, y_A, z_A)$ and ending at point $\vec{r}_B := (x_B, y_B, z_B)$:

$$[0,1] \ni u \mapsto \vec{r}(u) = (x_A(1-u) + x_B u, y_A(1-u) + y_B u, z_A(1-u) + z_B u); \tag{1.1}$$

  We denote it by $L_{A\to B}$.

- Two composed straight lines, starting at A, going through B, and ending at C:

$$[0,2] \ni u \mapsto \vec{r}(u) \in \mathbb{R}^3, \tag{1.2}$$

$$\vec{r}(u) = \begin{cases} (x_A(1-u) + x_B u, y_A(1-u) + y_B u, z_A(1-u) + z_B u), & u \in [0,1) \\ (x_B(2-u) + x_C(u-1), y_B(2-u) + y_C(u-1), z_B(2-u) + z_C(u-1)), & u \in [1,2] \end{cases}$$

  We denote it by $L_{A\to B\to C}$.

- A triangle starting at A, end ending at C:

$$[0,3] \ni u \mapsto \vec{r}(u) \in \mathbb{R}^3, \tag{1.3}$$

$$\vec{r}(u) = \begin{cases} (x_A(1-u) + x_B u, y_A(1-u) + y_B u, z_A(1-u) + z_B u), & u \in [0,1) \\ (x_B(2-u) + x_C(u-1), y_B(2-u) + y_C(u-1), z_B(2-u) + z_C(u-1)), & u \in [1,2) \\ (x_C(3-u) + x_A(u-2), y_C(3-u) + y_A(u-2), z_C(3-u) + z_A(u-2)), & u \in [2,3] \end{cases}$$

  We denote it by $L_{A\to B\to C\to A}$.

- A circle parallel to the xOy plane, with radius R and center at $\vec{r}_C = (x_C, y_C, z_C)$:

$$[0,2\pi] \ni u \mapsto \vec{r}(u) = (x_C + R\cos(u), y_C + R\sin(u), z_C); \tag{1.4}$$

  We denote it by $\mathcal{C}_R(\vec{r}_C)$.

It is very important to note that a path is always oriented, it has a starting and an ending point. $L_{A\to B}$ and $L_{B\to A}$ go through the same set of points, but have opposite orientation.

Given any path γ, we can construct the oppositely oriented path $\tilde{\gamma}$ by the formula:

$$[a,b] \ni u \mapsto \tilde{\gamma}(u) := \gamma(a+b-u). \tag{1.5}$$

A path $\gamma : [a,b] \to \mathbb{R}^3$ which has the property that $\gamma(a) = \gamma(b)$ is called a closed path.

1.2 Concatenation of paths

Assume that we have a path $\gamma_{A\to B}$ starting at point A and ending at B, and a path $\gamma_{B\to C}$ starting at B and ending at C. More precisely:

$$\gamma_{A\to B} : [a,b] \to \mathbb{R}^3, \quad \gamma_{A\to B}(a) = \vec{r}_A, \quad \gamma_{A\to B}(b) = \vec{r}_A,$$

and

$$\gamma_{B\to C} : [c,d] \to \mathbb{R}^3, \quad \gamma_{B\to C}(c) = \vec{r}_B, \quad \gamma_{B\to C}(d) = \vec{r}_C.$$

To concatenate $\gamma_{A\to B}$ and $\gamma_{B\to C}$ means to define a path $\gamma_{A\to C} := \gamma_{A\to B} \cup \gamma_{B\to C}$ which goes from A to C in the following way:

$$\gamma_{A\to C} : \left[\frac{a+c}{2}, \frac{b+d}{2}\right] \in \mathbb{R}^3, \tag{1.6}$$
$$\gamma_{A\to C}(u) = \begin{cases} \gamma_{A\to B}\left(a + (u - \frac{a+c}{2})\frac{4(b-a)}{b+d-a-c}\right), & u \in [\frac{a+c}{2}, \frac{a+b+c+d}{4}) \\ \gamma_{B\to C}\left(c + (u - \frac{a+b+c+d}{4})\frac{4(d-c)}{b+d-a-c}\right), & u \in [\frac{a+b+c+d}{4}, \frac{b+d}{2}] \end{cases}$$

If $a = c$ and $b = d$ then the formulas are much simpler:

$$\gamma_{A\to C} : [a,b] \in \mathbb{R}^3, \tag{1.7}$$
$$\gamma_{A\to C}(u) = \begin{cases} \gamma_{A\to B}(a + 2(u-a)), & u \in [a, \frac{a+b}{2}) \\ \gamma_{B\to C}\left(a + 2(u - \frac{a+b}{2})\right), & u \in [\frac{a+b}{2}, b] \end{cases}$$

1.3 The tangent vector field

Given a smooth path $\gamma : [a,b] \to \mathbb{R}^3$, $\gamma(u) = (x(u), y(u), z(u))$, we can compute its derivative at every point, thus obtaining its tangent vector field:

$$\gamma'(u) := (x'(u), y'(u), z'(u)). \tag{1.8}$$

For example, the tangent field for a straight line $L_{A\to B}$ is given by (see (1.1)):

$$\gamma'(u) := (x_B - x_A, y_B - y_A, z_B - z_A) = \vec{r}_B - \vec{r}_A, \tag{1.9}$$

and is constant. In the case of the circle $\mathcal{C}_R(\vec{r}_C)$ we have

$$\gamma'(u) := (-R\sin(u), R\cos(u), 0). \tag{1.10}$$

1.4 The length of a path

Remember that if we have two vectors $\vec{r}_1 = (x_1, y_1, z_1)$ and $\vec{r}_2 = (x_2, y_2, z_2)$, their dot product (scalar product) is defined as

$$\vec{r}_1 \cdot \vec{r}_2 = \langle \vec{r}_1, \vec{r}_2 \rangle = x_1x_2 + y_1y_2 + z_1z_2.$$

The length of a vector is

$$||\vec{r}|| := \sqrt{\vec{r}\cdot\vec{r}} = \sqrt{x^2+y^2+z^2}.$$

Consider $\gamma : [a,b] \to \mathbb{R}^3$, a smooth path. Its length is defined to be the integral:

$$\mathcal{L}(\gamma) := \int_a^b ||\gamma'(u)|| du. \tag{1.11}$$

1.5 Path integral of a vector field

Consider a vector field $\vec{F} : \mathbb{R}^3 \to \mathbb{R}^3$ given by $\vec{F}(\vec{r}) = (F_1(\vec{r}), F_2(\vec{r}), F_3(\vec{r}))$. Then the path integral of $\vec{F}$ on the path $\gamma : [a, b] \to \mathbb{R}^3$ is defined to be:

$$\int_\gamma \vec{F} \cdot d\gamma := \int_a^b \vec{F}(\gamma(u)) \cdot \gamma'(u) du. \tag{1.12}$$

One can show that the integral does not depend on the way we parametrize the path, as long as one keeps the same orientation. If γ is a closed path, then the integral is called circulation. Let us see two concrete examples, for the vector field $\vec{F}(\vec{r}) = (x - z, z, y + x)$.

1. First we compute $\int_{L_{A\to B}} \vec{F} \cdot d\gamma$.
   - compute $\gamma'(u)$. In this case it equals $\vec{r}_A - \vec{r}_B$;
   - compute the composed function $\vec{F}(\gamma(u))$. Here:

     $$\vec{F}(\gamma(u)) = ((x_A - z_A)(1-u) + (x_B - z_B)u, z_A(1-u) + z_B u, (y_A + x_A)(1-u) + (y_B + x_B)u);$$

   - compute the dot product $\vec{F}(\gamma(u)) \cdot \gamma'(u)$. Here it gives:

     $$(x_B - x_A)[(x_A - z_A)(1 - u) + (x_B - z_B)u] + (y_B - y_A)[z_A(1 - u) + z_B u]$$
     $$+(z_B - z_A)[(y_A + x_A)(1 - u) + (y_B + x_B)u].$$

   - integrate from a to b. Here we have:

     $$\frac{1}{2}(x_B - x_A)[(x_A - z_A) + (x_B - z_B)] + \frac{1}{2}(y_B - y_A)[z_A + z_B]$$
     $$+\frac{1}{2}(z_B - z_A)[(y_A + x_A) + (y_B + x_B)].$$

2. Second, let us compute the circulation of the same vector field on the circle $\mathcal{C}_R(\vec{r}_C)$.
   - compute $\gamma'(u)$. In this case it equals $(-R\sin(u), R\cos(u), 0)$;
   - compute the composed function $\vec{F}(\gamma(u))$. Here:

     $$\vec{F}(\gamma(u)) = (x_C + R\cos(u) - z_C, z_C, x_C + y_C + R\cos(u) + R\sin(u));$$

   - compute the dot product $\vec{F}(\gamma(u)) \cdot \gamma'(u)$. Here it gives:

     $$-R\sin(u)(x_C + R\cos(u) - z_C) + z_C R\cos(u).$$

   - integrate from 0 to 2π. Here the result is 0.

1.6 Important properties

We mention two important properties, given without proof. The first one says that if we integrate a vector field on the same path but in the opposite direction, then we get the same numerical value but with the opposite sign. More precisely (see (1.5)):

$$\int_\gamma \vec{F} \cdot d\gamma = -\int_{\tilde{\gamma}} \vec{F} \cdot d\tilde{\gamma}. \tag{1.13}$$

The second property says that if we integrate a vector field on a concatenated path, then the result is the sum of integrals on individual paths. In detail (see subsection 1.2):

$$\int_{\gamma_{A\to C}} \vec{F} \cdot d\gamma_{A\to C} = \int_{\gamma_{A\to B}} \vec{F} \cdot d\gamma_{A\to B} + \int_{\gamma_{B\to C}} \vec{F} \cdot d\gamma_{B\to C}. \tag{1.14}$$

Finally, let us look at the situation in which our vector field is the gradient of a given scalar function, that is

$$\vec{F}(\vec{r}) = \nabla V(\vec{r}) = [\partial_1 V(\vec{r}), \partial_1 V(\vec{r}), \partial_1 V(\vec{r})].$$

Then we can show that the path integral of $\vec{F}$ between two points A and B is independent of the path linking the two points. Indeed:

$$\begin{aligned}\int_\gamma \vec{F}\cdot d\gamma &= \int_a^b \nabla V(\gamma(t))\cdot\gamma'(t)dt = \int_a^b \left\{\frac{d}{dt}V(\gamma(t))\right\}dt = V(\gamma(b)) - V(\gamma(a))\\ &= V(\vec{r}_B) - V(\vec{r}_A).\end{aligned} \tag{1.15}$$

2 Surface integrals

2.1 General things about surfaces

Any surface in the three dimensional space is parametrized by two real variables. Let $D \subset \mathbb{R}^2$ denote the domain where these parameters live. Mathematically, a surface is the range of a function

$$D \ni (u,v) \mapsto \vec{r}(u,v) = (x(u,v), y(u,v), z(u,v)) \in \mathbb{R}^3.$$

Three important examples:

- Fix two vectors $\vec{r}_A := (x_A, y_A, z_A)$ and $\vec{r}_B := (x_B, y_B, z_B)$. The unique plane which contains both vectors is parametrized as:

  $$\mathbb{R}^2 \ni (u,v) \mapsto \vec{r}(u,v) := \vec{r}_A u + \vec{r}_B v = (x_A u + x_B v, y_A u + y_B v, z_A u + z_B v). \tag{2.1}$$

  Here $D = \mathbb{R}^2$.

- A sphere with center at $\vec{r}_S = (x_S, y_S, z_S)$ and radius R:

  $$\begin{aligned}&[0,\pi]\times[0,2\pi] \ni (\theta,\phi) \mapsto \vec{r}(\theta,\phi) \in \mathbb{R}^3,\\ &\vec{r}(\theta,\phi) = \vec{r}_S + R(\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)).\end{aligned} \tag{2.2}$$

  We denote it by $\partial B_R(\vec{r}_S)$. Here $D = [0,\pi]\times[0,2\pi]$.

- The disc contained in the circle $\mathcal{C}_R(\vec{r}_C)$ (see (1.4)):

  $$[0,R]\times[0,2\pi] \ni (\rho,\phi) \mapsto \vec{r}(\rho,\phi) = (x_C + \rho\cos(\phi), y_C + \rho\sin(\phi), z_C); \tag{2.3}$$

2.2 The infinitesimal surface element

Fix a point in D given by (u_0, v_0). If we vary (u,v) in a very small square around (u_0,v_0), then $\vec{r}(u,v)$ will cover a very small piece of our surface. Assume that $|u-u_0| = \delta u$ and $|v - v_0| = \delta v$ are small. Then this piece of surface can be approximated by a small portion of the tangent plane touching at $\vec{r}(u_0,v_0)$. Two vectors contained in this tangent plane are

$$\frac{\partial\vec{r}}{\partial u}(u_0,v_0), \quad \text{and} \quad \frac{\partial\vec{r}}{\partial v}(u_0,v_0). \tag{2.4}$$

The area of this surface element will approximately be:

$$d\sigma = \left|\frac{\partial\vec{r}}{\partial u}(u_0,v_0)\times\frac{\partial\vec{r}}{\partial v}(u_0,v_0)\right|\delta u\,\delta v. \tag{2.5}$$

We can also speak about orientation of surfaces. The above small surface element can be associated to the normal vector on the tangent plane, thus we can define a length one vector field

$$\vec{n}(u,v) := \frac{1}{\left|\frac{\partial\vec{r}}{\partial u}(u_0,v_0)\times\frac{\partial\vec{r}}{\partial v}(u_0,v_0)\right|}\frac{\partial\vec{r}}{\partial u}(u,v)\times\frac{\partial\vec{r}}{\partial v}(u,v). \tag{2.6}$$

If we swap u and v we obtain an opposite orientation. In case of closed surfaces, the normal is always taken in such a way that the normal vector points "out of the surface".

2.3 Integration formulas

Now assume that $f(\vec{r})$ is a scalar surface density of a certain physical quantity. Then this quantity is given by the integral:

$$\int_\sigma f(\vec{r})d\sigma := \int_D f(\vec{r}(u,v)) \left| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right| du\, dv. \tag{2.7}$$

If $\vec{F}$ is a vector field, then the flux of $\vec{F}$ through the oriented surface σ is defined to be:

$$\begin{aligned} \int_\sigma \vec{F}(\vec{r})d\vec{\sigma} &:= \int_D \vec{F}(\vec{r}(u,v)) \cdot \vec{n}(u,v) \left| \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right| du\, dv \\ &= \int_D \vec{F}(\vec{r}(u,v)) \cdot \left\{ \frac{\partial \vec{r}}{\partial u}(u,v) \times \frac{\partial \vec{r}}{\partial v}(u,v) \right\} du\, dv. \end{aligned} \tag{2.8}$$

2.4 Two examples

Let us go back to the disk defined in (2.3), and compute its normal vector field and the infinitesimal surface area. First we compute

$$\frac{\partial \vec{r}}{\partial \rho}(\rho,\phi) = (\cos(\phi), \sin(\phi), 0), \quad \frac{\partial \vec{r}}{\partial \phi}(\rho,\phi) = \rho(-\sin(\phi), \cos(\phi), 0).$$

These two vectors are orthogonal on each other, and moreover (see (2.6)):

$$\vec{n}(\rho,\phi) = (0,0,1).$$

The surface element is (see (2.5)):

$$d\sigma = \rho\, d\rho d\phi.$$

Let us consider a second example, i.e. the sphere in (2.2). In that case:

$$\frac{\partial \vec{r}}{\partial \theta}(\theta,\phi) = R(\cos(\theta)\cos(\phi), \cos(\theta)\sin(\phi), -\sin(\theta)), \quad \frac{\partial \vec{r}}{\partial \phi}(\theta,\phi) = R(-\sin(\theta)\sin(\phi), \sin(\theta)\cos(\phi), 0).$$

These two vectors are also orthogonal on each other, and moreover (see (2.6)):

$$\vec{n}(\theta,\phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta)) = \frac{\vec{r}(\theta,\phi)}{R}.$$

The surface element is (see (2.5)):

$$d\sigma = R^2 \sin(\theta)\, d\theta d\phi.$$