Chapter 4

Utility

Introduction. In the previous chapter, you learned about preferences and indifference curves. Here we study another way of describing preferences, the *utility function.* A utility function that represents a person's preferences is a function that assigns a utility number to each commodity bundle. The numbers are assigned in such a way that commodity bundle (x, y) gets a higher utility number than bundle (x', y') if and only if the consumer prefers (x, y) to (x', y'). If a consumer has the utility function $U(x_1, x_2)$, then she will be indifferent between two bundles if they are assigned the same utility.

If you know a consumer's utility function, then you can find the indifference curve passing through any commodity bundle. Recall from the previous chapter that when good 1 is graphed on the horizontal axis and good 2 on the vertical axis, the slope of the indifference curve passing through a point (x_1, x_2) is known as the *marginal rate of substitution.* An important and convenient fact is that *the slope of an indifference curve is minus the ratio of the marginal utility of good 1 to the marginal utility of good 2.* For those of you who know even a tiny bit of calculus, calculating marginal utilities is easy. To find the marginal utility of either good, you just take the derivative of utility with respect to the amount of that good, treating the amount of the other good as a constant. (If you don't know any calculus at all, you can calculate an approximation to marginal utility by the method described in your textbook. Also, at the beginning of this section of the workbook, we list the marginal utility functions for commonly encountered utility functions. Even if you can't compute these yourself, you can refer to this list when later problems require you to use marginal utilities.)

Example: Arthur's utility function is $U(x_1, x_2) = x_1 x_2$. Let us find the indifference curve for Arthur that passes through the point $(3, 4)$. First, calculate $U(3, 4) = 3 \times 4 = 12$. The indifference curve through this point consists of all (x_1, x_2) such that $x_1 x_2 = 12$. This last equation is equivalent to $x_2 = 12/x_1$. Therefore to draw Arthur's indifference curve through $(3, 4)$, just draw the curve with equation $x_2 = 12/x_1$. At the point (x_1, x_2), the marginal utility of good 1 is x_2 and the marginal utility of good 2 is x_1. Therefore Arthur's marginal rate of substitution at the point $(3, 4)$ is $-x_2/x_1 = -4/3$.

Example: Arthur's uncle, Basil, has the utility function $U^*(x_1, x_2) = 3x_1 x_2 - 10$. Notice that $U^*(x_1, x_2) = 3U(x_1, x_2) - 10$, where $U(x_1, x_2)$ is Arthur's utility function. Since U^* is a positive multiple of U minus a constant, it must be that any change in consumption that increases U will also increase U^* (and vice versa). Therefore we say that Basil's utility function is a *monotonic increasing transformation* of Arthur's utility function. Let us find Basil's indifference curve through the point $(3, 4)$. First we find that $U^*(3, 4) = 3 \times 3 \times 4 - 10 = 26$. The indifference curve passing through this point consists of all (x_1, x_2) such that $3x_1x_2 - 10 = 26$. Simplify this last expression by adding 10 to both sides of the equation and dividing both sides by 3. You find $x_1x_2 = 12$, or equivalently, $x_2 = 12/x_1$. This is exactly the same curve as Arthur's indifference curve through $(3, 4)$. We could have known in advance that this would happen, because if two consumers' utility functions are monotonic increasing transformations of each other, then these consumers must have the same preference relation between any pair of commodity bundles.

When you have finished this workout, we hope that you will be able to do the following:

- Draw an indifference curve through a specified commodity bundle when you know the utility function.
- Calculate marginal utilities and marginal rates of substitution when you know the utility function.
- Determine whether one utility function is just a "monotonic transformation" of another and know what that implies about preferences.
- Find utility functions that represent preferences when goods are perfect substitutes and when goods are perfect complements.
- Recognize utility functions for commonly studied preferences such as perfect substitutes, perfect complements, and other kinked indifference curves, quasilinear utility, and Cobb-Douglas utility.

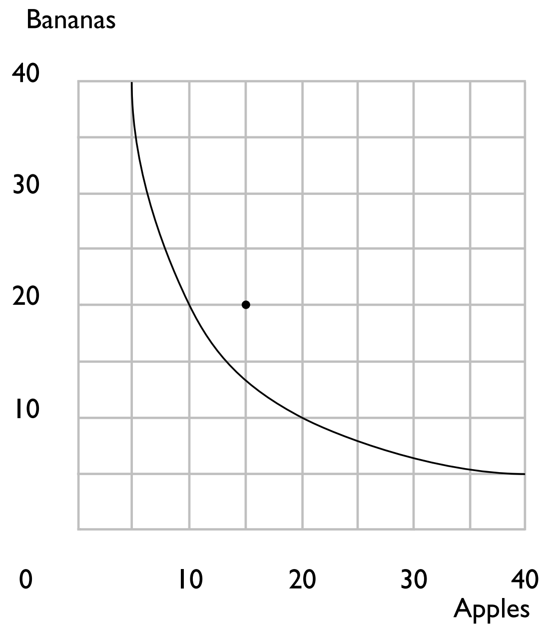
4.0 Warm Up Exercise. This is the first of several "warm up exercises" that you will find in *Workouts*. These are here to help you see how to do calculations that are needed in later problems. The answers to *all* warm up exercises are in your answer pages. If you find the warm up exercises easy and boring, go ahead—skip them and get on to the main problems. You can come back and look at them if you get stuck later.

This exercise asks you to calculate marginal utilities and marginal rates of substitution for some common utility functions. These utility functions will reappear in several chapters, so it is a good idea to get to know them now. If you know calculus, you will find this to be a breeze. Even if your calculus is shaky or nonexistent, you can handle the first three utility functions just by using the definitions in the textbook. These three are easy because the utility functions are linear. If you do not know any calculus, fill in the rest of the answers from the back of the workbook and keep a copy of this exercise for reference when you encounter these utility functions in later problems.

$u(x_1, x_2)$	$MU_1(x_1, x_2)$	$MU_2(x_1, x_2)$	$MRS(x_1, x_2)$
$2x_1 + 3x_2$	2	3	$-2/3$
$4x_1 + 6x_2$	4	6	$-2/3$
$ax_1 + bx_2$	a	b	$-a/b$
$2\sqrt{x_1} + x_2$	$\frac{1}{\sqrt{x_1}}$	1	$-\frac{1}{\sqrt{x_1}}$
$\ln x_1 + x_2$	$1/x_1$	1	$-1/x_1$
$v(x_1) + x_2$	$v'(x_1)$	1	$-v'(x_1)$
$x_1 x_2$	x_2	x_1	$-x_2/x_1$
$x_1^a x_2^b$	$ax_1^{a-1}x_2^b$	$bx_1^a x_2^{b-1}$	$-\frac{ax_2}{bx_1}$
$(x_1 + 2)(x_2 + 1)$	$x_2 + 1$	$x_1 + 2$	$-\left(\frac{x_2+1}{x_1+2}\right)$
$(x_1 + a)(x_2 + b)$	$x_2 + b$	$x_1 + a$	$-\left(\frac{x_2+b}{x_1+a}\right)$
$x_1^a + x_2^a$	ax_1^{a-1}	ax_2^{a-1}	$-\left(\frac{x_1}{x_2}\right)^{a-1}$

4.1 (0) Remember Charlie from Chapter 3? Charlie consumes apples and bananas. We had a look at two of his indifference curves. In this problem we give you enough information so you can find *all* of Charlie's indifference curves. We do this by telling you that Charlie's utility function happens to be $U(x_A, x_B) = x_A x_B$.

(a) Charlie has 40 apples and 5 bananas. Charlie's utility for the bundle $(40, 5)$ is $U(40, 5) =$ 200. The indifference curve through $(40, 5)$ includes all commodity bundles (x_A, x_B) such that $x_A x_B =$ 200. So the indifference curve through $(40, 5)$ has the equation $x_B = \frac{200}{x_A}$. On the graph below, draw the indifference curve showing all of the bundles that Charlie likes exactly as well as the bundle $(40, 5)$.

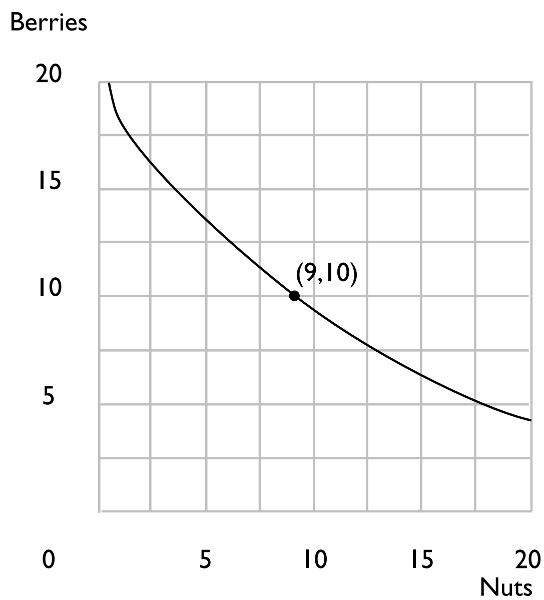


(b) Donna offers to give Charlie 15 bananas if he will give her 25 apples. Would Charlie have a bundle that he likes better than $(40, 5)$ if he makes this trade? Yes. What is the largest number of apples that Donna could demand from Charlie in return for 15 bananas if she expects him to be willing to trade or at least indifferent about trading? 30. (Hint: If Donna gives Charlie 15 bananas, he will have a total of 20 bananas. If he has 20 bananas, how many apples does he need in order to be as well-off as he would be without trade?)

4.2 (0) Ambrose, whom you met in the last chapter, continues to thrive on nuts and berries. You saw two of his indifference curves. One indifference curve had the equation $x_2 = 20 - 4\sqrt{x_1}$, and another indifference curve had the equation $x_2 = 24 - 4\sqrt{x_1}$, where x_1 is his consumption of nuts and x_2 is his consumption of berries. Now it can be told that Ambrose has quasilinear utility. In fact, his preferences can be represented by the utility function $U(x_1, x_2) = 4\sqrt{x_1} + x_2$.

(a) Ambrose originally consumed 9 units of nuts and 10 units of berries. His consumption of nuts is reduced to 4 units, but he is given enough berries so that he is just as well-off as he was before. After the change, how many units of berries does Ambrose consume? 14.

(b) On the graph below, indicate Ambrose's original consumption and sketch an indifference curve passing through this point. As you can verify, Ambrose is indifferent between the bundle (9,10) and the bundle (25,2). If you doubled the amount of each good in each bundle, you would have bundles (18,20) and (50,4). Are these two bundles on the same indifference curve? No. (Hint: How do you check whether two bundles are indifferent when you know the utility function?)



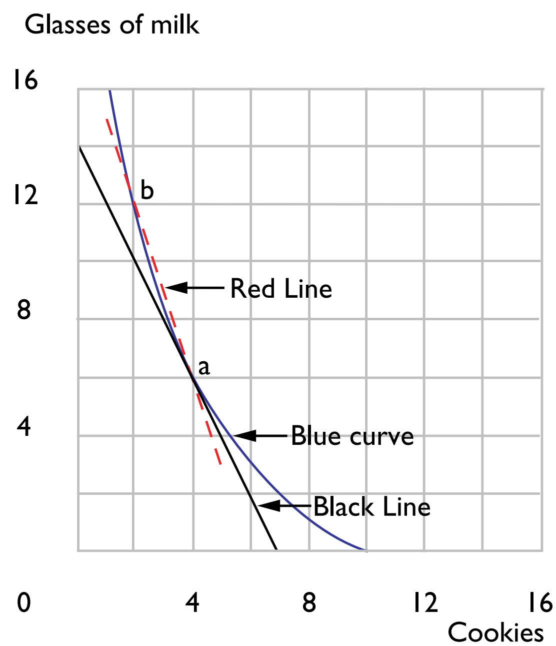
(c) What is Ambrose's marginal rate of substitution, $MRS(x_1, x_2)$, when he is consuming the bundle $(9, 10)$? (Give a numerical answer.) $-2/3$. What is Ambrose's marginal rate of substitution when he is consuming the bundle $(9, 20)$? $-2/3$.

(d) We can write a general expression for Ambrose's marginal rate of substitution when he is consuming commodity bundle (x_1, x_2). This is $MRS(x_1, x_2) =$ $-2/\sqrt{x_1}$. Although we always write $MRS(x_1, x_2)$ as a function of the two variables, x_1 and x_2, we see that Ambrose's utility function has the special property that his marginal rate of substitution does not change when the variable x_2 changes.

4.3 (0) Burt's utility function is $U(x_1, x_2) = (x_1 + 2)(x_2 + 6)$, where x_1 is the number of cookies and x_2 is the number of glasses of milk that he consumes.

(a) What is the slope of Burt's indifference curve at the point where he is consuming the bundle $(4, 6)$? -2. Use pencil or black ink to draw a line with this slope through the point $(4, 6)$. (Try to make this graph fairly neat and precise, since details will matter.) The line you just drew is the *tangent line* to the consumer's indifference curve at the point $(4, 6)$.

(b) The indifference curve through the point $(4, 6)$ passes through the points (10 ,0), (7, 2), and (2, 12). Use blue ink to sketch in this indifference curve. Incidentally, the equation for Burt's indifference curve through the point $(4, 6)$ is $x_2 =$ $72/(x_1 + 2) - 6$.



(c) Burt currently has the bundle $(4, 6)$. Ernie offers to give Burt 9 glasses of milk if Burt will give Ernie 3 cookies. If Burt makes this trade, he would have the bundle $(1, 15)$. Burt refuses to trade. Was this a wise decision? Yes, $U(1, 15) = 63 < U(4, 6) = 72$. Mark the bundle $(1, 15)$ on your graph.

(d) Ernie says to Burt, "Burt, your marginal rate of substitution is -2. That means that an extra cookie is worth only twice as much to you as an extra glass of milk. I offered to give you 3 glasses of milk for every cookie you give me. If I offer to give you more than your marginal rate of substitution, then you should want to trade with me." Burt replies,

"Ernie, you are right that my marginal rate of substitution is -2. That means that I am willing to make *small* trades where I get more than 2 glasses of milk for every cookie I give you, but 9 glasses of milk for 3 cookies is too big a trade. My indifference curves are not straight lines, you see." Would Burt be willing to give up 1 cookie for 3 glasses of milk? Yes, $U(3,9) = 75 > U(4,6) = 72$. Would Burt object to giving up 2 cookies for 6 glasses of milk? No, $U(2,12) = 72 = U(4,6)$.

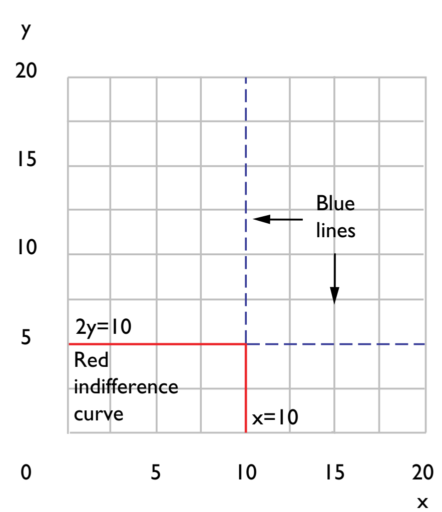
(e) On your graph, use red ink to draw a line with slope -3 through the point $(4,6)$. This line shows all of the bundles that Burt can achieve by trading cookies for milk (or milk for cookies) at the rate of 1 cookie for every 3 glasses of milk. Only a segment of this line represents trades that make Burt better off than he was without trade. Label this line segment on your graph AB.

4.4 (0) Phil Rupp's utility function is $U(x, y) = \max\{x, 2y\}$.

(a) On the graph below, use blue ink to draw and label the line whose equation is $x = 10$. Also use blue ink to draw and label the line whose equation is $2y = 10$.

(b) If $x = 10$ and $2y < 10$, then $U(x, y) =$ 10. If $x < 10$ and $2y = 10$, then $U(x, y) =$ 10.

(c) Now use red ink to sketch in the indifference curve along which $U(x, y) = 10$. Does Phil have convex preferences? No.



4.5 (0) As you may recall, Nancy Lerner is taking Professor Stern's economics course. She will take two examinations in the course, and her score for the course is the minimum of the scores that she gets on the two exams. Nancy wants to get the highest possible score for the course.

(a) Write a utility function that represents Nancy's preferences over alternative combinations of test scores x_1 and x_2 on tests 1 and 2 respectively. $U(x_1, x_2) =$ $\min\{x_1, x_2\}$, or any monotonic transformation.

4.6 (0) Remember Shirley Sixpack and Lorraine Quiche from the last chapter? Shirley thinks a 16-ounce can of beer is just as good as two 8-ounce cans. Lorraine only drinks 8 ounces at a time and hates stale beer, so she thinks a 16-ounce can is no better or worse than an 8-ounce can.

(a) Write a utility function that represents Shirley's preferences between commodity bundles comprised of 8-ounce cans and 16-ounce cans of beer. Let X stand for the number of 8-ounce cans and Y stand for the number of 16-ounce cans. $u(X, Y) = X + 2Y$.

(b) Now write a utility function that represents Lorraine's preferences.

$u(X, Y) = X + Y$.

(c) Would the function utility $U(X, Y) = 100X + 200Y$ represent Shirley's preferences? Yes. Would the utility function $U(x, y) = (5X + 10Y)^2$ represent her preferences? Yes. Would the utility function $U(x, y) = X + 3Y$ represent her preferences? No.

(d) Give an example of two commodity bundles such that Shirley likes the first bundle better than the second bundle, while Lorraine likes the second bundle better than the first bundle. Shirley prefers (0,2) to (3,0). Lorraine disagrees.

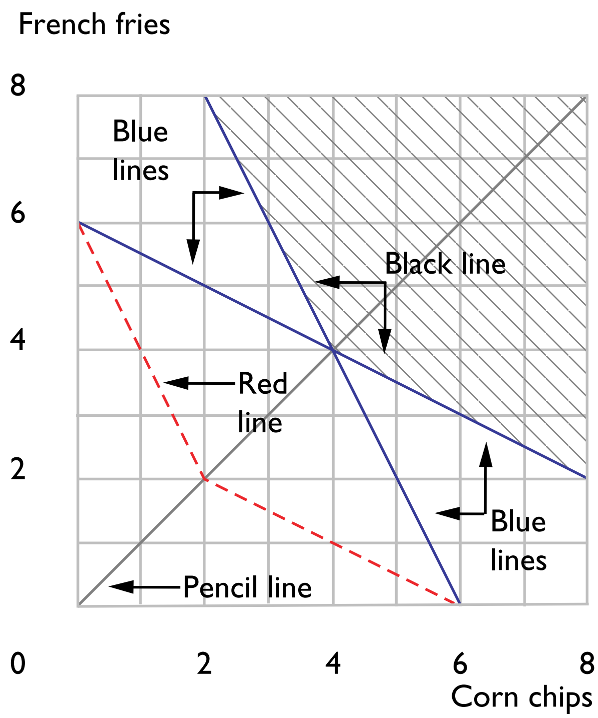
4.7 (0) Harry Mazzola has the utility function $u(x_1, x_2) = \min\{x_1 + 2x_2, 2x_1 + x_2\}$, where x_1 is his consumption of corn chips and x_2 is his consumption of french fries.

(a) On the graph below, use a pencil to draw the locus of points along which $x_1 + 2x_2 = 2x_1 + x_2$. Use blue ink to show the locus of points for which $x_1 + 2x_2 = 12$, and also use blue ink to draw the locus of points for which $2x_1 + x_2 = 12$.

(b) On the graph you have drawn, shade in the region where *both* of the following inequalities are satisfied: $x_1 + 2x_2 \geq 12$ and $2x_1 + x_2 \geq 12$.

At the bundle $(x_1, x_2) = (8, 2)$, one sees that $2x_1 + x_2 =$ 18 and $x_1 + 2x_2 =$ 12. Therefore $u(8, 2) =$ 12.

(c) Use black ink to sketch in the indifference curve along which Harry's utility is 12. Use red ink to sketch in the indifference curve along which Harry's utility is 6. (Hint: Is there anything about Harry Mazzola that reminds you of Mary Granola?)



(d) At the point where Harry is consuming 5 units of corn chips and 2 units of french fries, how many units of corn chips would he be willing to trade for one unit of french fries? 2.

4.8 (1) Vanna Boogie likes to have large parties. She also has a strong preference for having exactly as many men as women at her parties. In fact, Vanna's preferences among parties can be represented by the utility function $U(x, y) = \min\{2x - y, 2y - x\}$ where x is the number of women and y is the number of men at the party. On the graph below, let us try to draw the indifference curve along which Vanna's utility is 10.

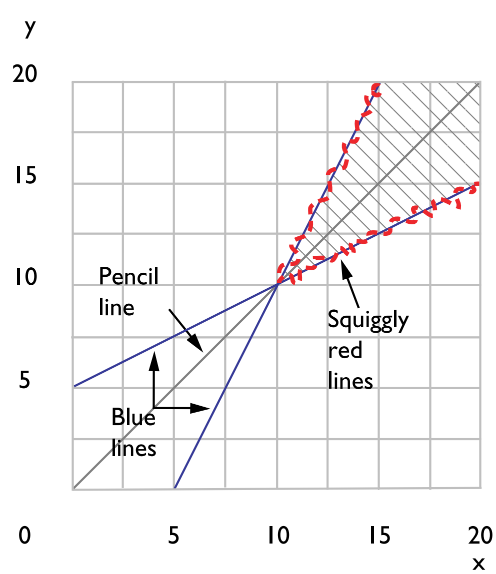
(a) Use pencil to draw the locus of points at which $x = y$. What point on this gives Vanna a utility of 10? $(10, 10)$. Use blue ink to draw the line along which $2y - x = 10$. When $\min\{2x - y, 2y - x\} = 2y - x$,

there are (more men than women, more women than men)? `More women.` Draw a squiggly red line over the part of the blue line for which $U(x,y) = \min\{2x - y, 2y - x\} = 2y - x$. This shows all the combinations that Vanna thinks are just as good as $(10,10)$ but where there are (more men than women, more women than men)? `More women.` Now draw a blue line along which $2x - y = 10$. Draw a squiggly red line over the part of this new blue line for which $\min\{2x - y, 2y - x\} = 2x - y$. Use pencil to shade in the area on the graph that represents all combinations that Vanna likes at least as well as $(10,10)$.

(b) Suppose that there are 9 men and 10 women at Vanna's party. Would Vanna think it was a better party or a worse party if 5 more men came to her party? `Worse.`

(c) If Vanna has 16 women at her party and more men than women, and if she thinks the party is exactly as good as having 10 men and 10 women, how many men does she have at the party? `22.` If Vanna has 16 women at her party and more women than men, and if she thinks the party is exactly as good as having 10 men and 10 women, how many men does she have at her party? `13.`

(d) Vanna's indifference curves are shaped like what letter of the alphabet? `V.`



4.9 (0) Suppose that the utility functions $u(x,y)$ and $v(x,y)$ are related by $v(x,y) = f(u(x,y))$. In each case below, write "Yes" if the function f is a positive monotonic transformation and "No" if it is not. (Hint for calculus users: A differentiable function $f(u)$ is an increasing function of u if its derivative is positive.)

(a) $f(u) = 3.141592u$. Yes.

(b) $f(u) = 5,000 - 23u$. No.

(c) $f(u) = u - 100,000$. Yes.

(d) $f(u) = \log_{10} u$. Yes.

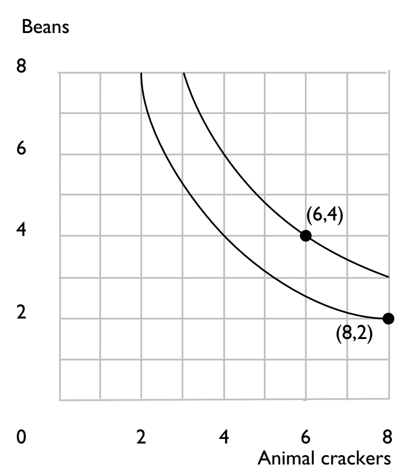
(e) $f(u) = -e^{-u}$. Yes.

(f) $f(u) = 1/u$. No.

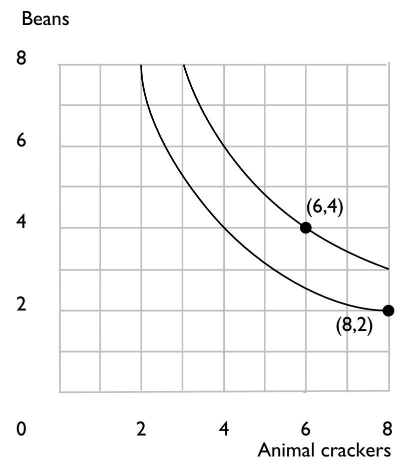
(g) $f(u) = -1/u$. Yes.

4.10 (0) Martha Modest has preferences represented by the utility function $U(a, b) = ab/100$, where a is the number of ounces of animal crackers that she consumes and b is the number of ounces of beans that she consumes.

(a) On the graph below, sketch the locus of points that Martha finds indifferent to having 8 ounces of animal crackers and 2 ounces of beans. Also sketch the locus of points that she finds indifferent to having 6 ounces of animal crackers and 4 ounces of beans.



(b) Bertha Brassy has preferences represented by the utility function $V(a, b) = 1{,}000a^2b^2$, where a is the number of ounces of animal crackers that she consumes and b is the number of ounces of beans that she consumes. On the graph below, sketch the locus of points that Bertha finds indifferent to having 8 ounces of animal crackers and 2 ounces of beans. Also sketch the locus of points that she finds indifferent to having 6 ounces of animal crackers and 4 ounces of beans.



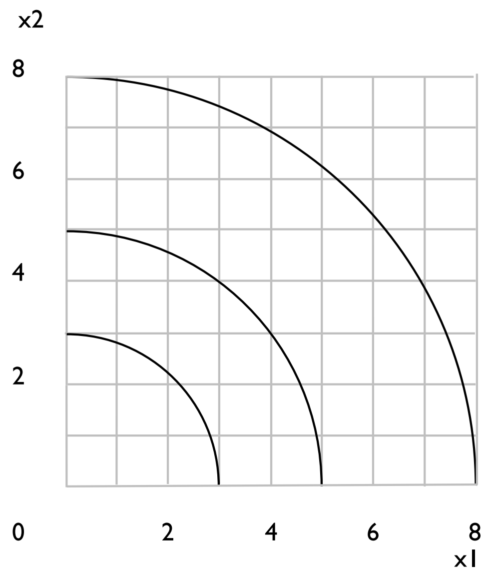
(c) Are Martha's preferences convex? `Yes.` Are Bertha's? `Yes.`

(d) What can you say about the difference between the indifference curves you drew for Bertha and those you drew for Martha? `There is no difference.`

(e) How could you tell this was going to happen without having to draw the curves? `Their utility functions only differ by a monotonic transformation.`

4.11 (0) Willy Wheeler's preferences over bundles that contain non-negative amounts of x_1 and x_2 are represented by the utility function $U(x_1, x_2) = x_1^2 + x_2^2$.

(a) Draw a few of his indifference curves. What kind of geometric figure are they? `Quarter circles centered at the origin.` Does Willy have convex preferences? `No.`



Calculus **4.12 (0)** Joe Bob has a utility function given by $u(x_1, x_2) = x_1^2 + 2x_1x_2 + x_2^2$.

(a) Compute Joe Bob's marginal rate of substitution: $MRS(x_1, x_2) =$

-1.

(b) Joe Bob's straight cousin, Al, has a utility function $v(x_1, x_2) = x_2 + x_1$. Compute Al's marginal rate of substitution. $MRS(x_1, x_2) =$ -1.

(c) Do $u(x_1, x_2)$ and $v(x_1, x_2)$ represent the same preferences? Yes. Can you show that Joe Bob's utility function is a monotonic transformation of Al's? (Hint: Some have said that Joe Bob is square.) Notice that $u(x_1, x_2) = [v(x_1, x_2)]^2$.

4.13 (0) The idea of assigning numerical values to determine a preference ordering over a set of objects is not limited in application to commodity bundles. The *Bill James Baseball Abstract* argues that a baseball player's batting average is not an adequate measure of his offensive productivity. Batting averages treat singles just the same as extra base hits. Furthermore they do not give credit for "walks," although a walk is almost as good as a single. James argues that a double in two at-bats is better than a single, but not as good as two singles. To reflect these considerations, James proposes the following index, which he calls "runs created." Let A be the number of hits plus the number of walks that a batter gets in a season. Let B be the number of total bases that the batter gets in the season. (Thus, if a batter has S singles, W walks, D doubles, T triples, and H

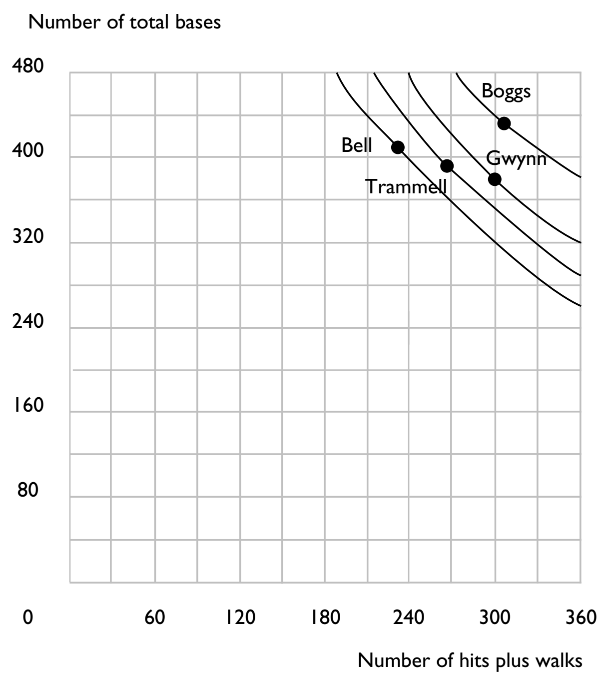
home runs, then $A = S+D+T+H+W$ and $B = S+W+2D+3T+4H$.) Let N be the number of times the batter bats. Then his index of runs created in the season is defined to be AB/N and will be called his RC.

(a) In 1987, George Bell batted 649 times. He had 39 walks, 105 singles, 32 doubles, 4 triples, and 47 home runs. In 1987, Wade Boggs batted 656 times. He had 105 walks, 130 singles, 40 doubles, 6 triples, and 24 home runs. In 1987, Alan Trammell batted 657 times. He had 60 walks, 140 singles, 34 doubles, 3 triples, and 28 home runs. In 1987, Tony Gwynn batted 671 times. He had 82 walks, 162 singles, 36 doubles, 13 triples, and 7 home runs. We can calculate A, the number of hits plus walks, B the number of total bases, and RC, the runs created index for each of these players. For Bell, $A = 227$, $B = 408$, $RC = 143$. For Boggs, $A = 305$, $B = 429$, $RC = 199$. For Trammell, $A = 265$, $B = 389$, $RC = 157$. For Gwynn, $A =$ 300 , $B =$ 383 , $RC =$ 171.

(b) If somebody has a preference ordering among these players, based only on the runs-created index, which player(s) would she prefer to Trammell?

Boggs and Gwynn.

(c) The differences in the number of times at bat for these players are small, and we will ignore them for simplicity of calculation. On the graph below, plot the combinations of A and B achieved by each of the players. Draw four "indifference curves," one through each of the four points you have plotted. These indifference curves should represent combinations of A and B that lead to the same number of runs-created.



4.14 (0) This problem concerns the runs-created index discussed in the preceding problem. Consider a batter who bats 100 times and always either makes an out, hits for a single, or hits a home run.

(a) Let x be the number of singles and y be the number of home runs in 100 at-bats. Suppose that the utility function $U(x, y)$ by which we evaluate alternative combinations of singles and home runs is the runs-created index. Then the formula for the utility function is $U(x, y) =$

$(x + y)(x + 4y)/100$.

(b) Let's try to find out about the shape of an indifference curve between singles and home runs. Hitting 10 home runs and no singles would give him the same runs-created index as hitting 20 singles and no home runs. Mark the points $(0, 10)$ and $(x, 0)$, where $U(x, 0) = U(0, 10)$.

(c) Where x is the number of singles you solved for in the previous part, mark the point $(x/2, 5)$ on your graph. Is $U(x/2, 5)$ greater than or less than or equal to $U(0, 10)$? Greater than. Is this consistent with the batter having convex preferences between singles and home runs?

Yes.