**AALBORG UNIVERSITET**
**LANDINSPEKTØRUDDANNELSEN - 4.sem.**
**Matematisk grundlag for kortprojektioner**
**Dato: May 2000**

# Mercator and Transversal Mercator projection

## 1 Mercator Projection

The Mercator projection is probably the best known projection of them all. It was first constructed by Gerardus Krämer (Mercator), a Flemish cartographer, who lived in the period 1512-94. His intention was to construct a map which was good for navigation.

We will describe it as an example of a conformal projection, and the reader interested in Mercators original problem should consult for instance [3].

Remember, that a *conformal* map from (a subset of) the sphere (or the ellipsoid ) to the plane is - in terms of geographical coordinates - a map $\mathbf{x}(\lambda, \varphi)$ from (a subset of) the rectangle $R = ]-\pi, \pi[\times]-\pi/2, \pi/2[$ to the plane $\mathbb{R}^2$ satisfying the partial differential equations

$$\begin{aligned} \frac{\tilde{E}}{E} &= \frac{\tilde{G}}{G} \\ \tilde{F} &= 0 \end{aligned} \tag{1}$$

where $\tilde{E}$, $\tilde{F}$ and $\tilde{G}$ are the coefficients of the first fundamental form of $\mathbf{x}$, and $E, F$ and $G$ are the coefficients of the first fundamental form of the sphere (or the ellipsoid).

A general *cylindrical projection* maps parallels into horizontal and meridians into vertical straight lines in the plane. Moreover, we require that two pairs of meridians differing by the same degree have to be mapped to horizontal straight lines differing by the same distance. In other words, we want the map $\mathbf{x} : R \to \mathbb{R}^2$ to be of the form $\mathbf{x}(\lambda, \varphi) = (c\lambda, h(\varphi))$, where $c$ is a constant.

Furthermore, we assume that $c$ is positive and that $h$ is strictly increasing. The assumption $c > 0$ is to assure that meridians with $\lambda > 0$ get mapped to vertical lines on the right hand side of the Greenwich Meridian $\lambda = 0$, and those with $\lambda < 0$ go to the left of the Greenwich meridian. If $h$ is strictly increasing, a parallel $B_{\varphi_1}$ which is north of a parallel $B_{\varphi_2}$ will be mapped to a horizontal line *over* that corresponding to $\varphi_2$.

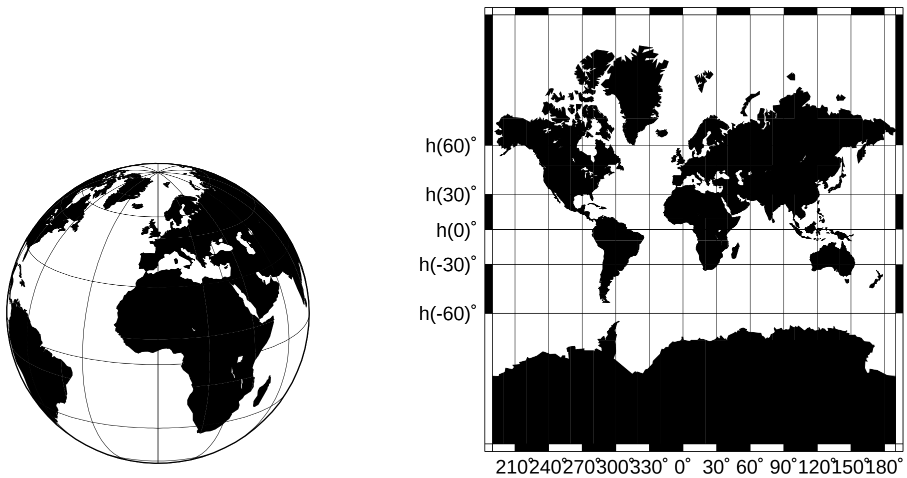


Figure 1: The cylinderprojection $(\lambda, \varphi) \to (\lambda, h(\varphi))$

Since we are looking for conformal projections, the basic question is: (How) can one choose $h$, such that $\mathbf{x}$ will give rise to a conformal cylindrical projection? In other words: Solve the partial differential equations (1) in this class of projections.

## 1.1 Mercator projection from the sphere

To this end, we have to establish the first fundamental forms of $\mathbf{x}$ and of the sphere of radius $R$ (with geographical coordinates ), i.e.,

$$\begin{aligned} \tilde{E} &= \frac{\partial \mathbf{x}}{\partial \lambda} \cdot \frac{\partial \mathbf{x}}{\partial \lambda} = \begin{pmatrix} c \\ 0 \end{pmatrix} \cdot \begin{pmatrix} c \\ 0 \end{pmatrix} = c^2, \\ \tilde{G} &= \frac{\partial \mathbf{x}}{\partial \varphi} \cdot \frac{\partial \mathbf{x}}{\partial \varphi} = \begin{pmatrix} 0 \\ h'(\varphi) \end{pmatrix} \cdot \begin{pmatrix} 0 \\ h'(\varphi) \end{pmatrix} = (h'(\varphi))^2, \\ \tilde{F} &= \frac{\partial \mathbf{x}}{\partial \lambda} \cdot \frac{\partial \mathbf{x}}{\partial \varphi} = 0; \end{aligned} \tag{2}$$

$$\begin{aligned} E &= R^2 \cos^2 \varphi, \\ F &= 0, \\ G &= R^2. \end{aligned} \tag{3}$$

Hence, a conformal cylindrical projection from the sphere has to satisfy the differential equation

$$\frac{\tilde{E}}{E} = \frac{\tilde{G}}{G} \Leftrightarrow (\frac{c}{R\cos\varphi})^2 = (\frac{h'(\varphi)}{R})^2 \Leftrightarrow h'(\varphi) = \pm\frac{c}{\cos\varphi}.$$

Since we assume that $h$ is increasing and $c$ is positive, the sign is positive, i.e.,

$$h'(\varphi) = \frac{c}{\cos\varphi} \tag{4}$$

By integration, one may identify the functions $h$ satisfying 4 as

$$h(\varphi) = c\ln(\tan(\frac{\pi}{4} + \frac{\varphi}{2})) + k$$

where $k$ is a constant. For $k = 0$, the image of the Equator corresponds to the first axis of the coordinate system. The reader is asked to differentiate $h$ and to check 4. We have proven:

**Theorem 1.1** *Every conformal cylindrical projection from the sphere is of the form*

$$\mathbf{x}(\lambda, \varphi) = (c\lambda, c\ln(\tan(\frac{\pi}{4} + \frac{\varphi}{2})))$$

Since $\mathbf{x}$ is a conformal projection, the measure is independent of direction and given as:

$$m(\lambda, \varphi) = \sqrt{\frac{e}{E}} = \frac{c}{\cos\varphi}$$

## 1.2 Mercator projection from the ellipsoid

If we approximate the Earth by an ellipsoid instead of a sphere, we can still define conformal cylindrical projections in the same way.

The first fundamental form of the ellipsoid of revolution with semiaxis $a$ and $b$ is calculated as follows

$$E = N^2\cos^2\varphi,$$

$$F = 0,$$

$$G = M^2,$$

where

$$N = \frac{a^2}{\sqrt{a^2\cos^2\varphi + b^2\sin^2\varphi}}$$

and

$$M = \frac{a^2b^2}{(a^2\cos^2\varphi + b^2\sin^2\varphi)^{3/2}}$$

The differential equation determining $h(\varphi)$ in this case is:

$$\frac{c}{N\cos\varphi} = \frac{h'(\varphi)}{M}$$

Hence we have to find a function $h$ such that

$$h'(\varphi) = \frac{cM}{N\cos\varphi}$$

Integration of the latter function is subtle and yields a complicated solution:

$$h(\varphi) = c\ln(\tan(\frac{\pi}{4} + \frac{\varphi}{2})(\frac{1-\epsilon\sin\varphi}{1+\epsilon\sin\varphi})^{\epsilon/2})$$

where $\epsilon = \sqrt{1 - \frac{b^2}{a^2}}$ is the excentricity of the ellipse.

The measure is in this case given as:

$$m(\lambda, \varphi) = \sqrt{\frac{e}{E}} = \frac{c}{N\cos\varphi}$$

# 2 Transversal Mercator projection

## 2.1 The sphere

Having found a particular class of conformal maps - Mercator projections -, one is tempted to reuse it to get new ones, just by changing the position of the projection surface. The Transversal Mercator projection is constructed in this manner by placing the cylinder in such a way that it touches the sphere along a meridian with longitude $\lambda_0$ (and $\pi - \lambda_0$) as shown below Fig. 2, and then applying the Mercator map.

We want to find a formula for the map $\mathbf{x}(\lambda, \varphi)$ corresponding to this procedure. This is done in two steps: First, we rotate the sphere in such a way that the $\lambda_0$ meridan plane is mapped into the Equatorial plane. Then, we apply the normal Mercator projection to the rotated sphere. See Fig. 2.1

Let $e_1$, $e_2$, $e_3$ denote the standard basis in $\mathbb{R}^3$, and let

$$e_1' = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, e_2' = \begin{pmatrix} \cos\lambda_0 \\ \sin\lambda_0 \\ 0 \end{pmatrix}, e_3' = \begin{pmatrix} -\sin\lambda_0 \\ \cos\lambda_0 \\ 0 \end{pmatrix}$$

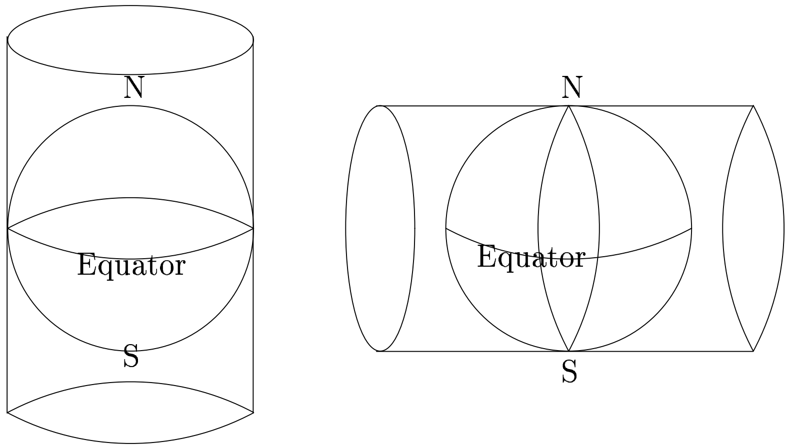


Figure 2: Rotating the cylinder.

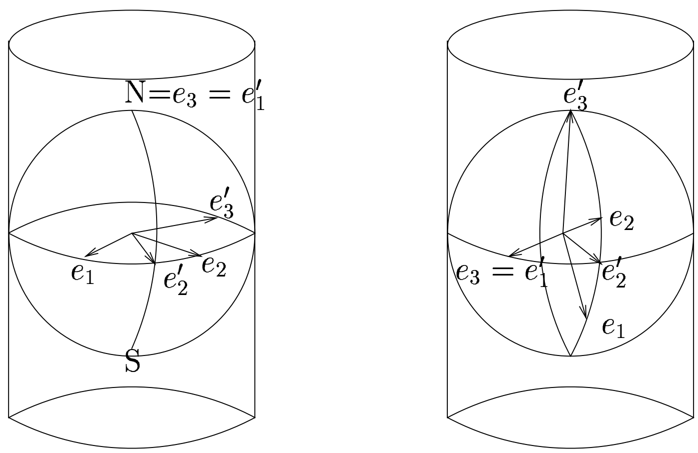


Figure 3: Rotating the sphere.

The rotation is a linear map which sends $e_1'$ to $e_1$, $e_2'$ to $e_2$ and $e_3'$ to $e_3$.

Let $A$ denote the $3 \times 3$ matrix describing this map. Then we can write down the inverse $A^{-1}$ immediately, since it maps $e_1$ to $e_1'$, $e_2$ to $e_2'$ and $e_3$ to $e_3'$, i.e.,

$$A^{-1} = \begin{pmatrix} 0 & \cos\lambda_0 & -\sin\lambda_0 \\ 0 & \sin\lambda_0 & \cos\lambda_0 \\ 1 & 0 & 0 \end{pmatrix}$$

Since $A$ and $A^{-1}$ are orthogonal, we can determine $A$ by

$$A = (A^{-1})^{-1} = (A^{-1})^t = \begin{pmatrix} 0 & 0 & 1 \\ \cos\lambda_0 & \sin\lambda_0 & 0 \\ -\sin\lambda_0 & \cos\lambda_0 & 0 \end{pmatrix}$$

This gives us the rotation in terms of Cartesian coordinates $(x, y, z)$. In fact we need the geographical coordinates $(\lambda', \varphi')$ of the image of a point with geographical coordinates $(\lambda, \varphi)$.

For

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{pmatrix},$$

we obtain

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A \begin{pmatrix} \cos\varphi\cos\lambda \\ \cos\varphi\sin\lambda \\ \sin\varphi \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ \cos\varphi\cos(\lambda-\lambda_0) \\ \cos\varphi\sin(\lambda-\lambda_0) \end{pmatrix},$$

using the trigonometric formulae for differences of angles. Hence

$$\begin{pmatrix} \cos\varphi'\cos\lambda' \\ \cos\varphi'\sin\lambda' \\ \sin\varphi' \end{pmatrix} = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \sin\varphi \\ \cos\varphi\cos(\lambda-\lambda_0) \\ \cos\varphi\sin(\lambda-\lambda_0) \end{pmatrix}.$$

In particular

$$\sin\varphi' = z' = \cos\varphi\sin(\lambda-\lambda_0),$$

and

$$\tan\lambda' = \frac{y'}{x'} = \cot\varphi\cos(\lambda-\lambda_0).$$

Solving for $\lambda'$ and $\varphi'$, we find

$$\lambda' = \arctan(\cot\varphi\cos(\lambda-\lambda_0)),$$
$$\varphi' = \arcsin(\cos\varphi\sin(\lambda-\lambda_0).)$$

A formula for the Transversal Mercator projection is then derived as follows:

$$(\lambda,\varphi) \to (\lambda',\varphi') \to (c\lambda', c\ln(\tan(\frac{\pi}{4}+\frac{\varphi'}{2}))) = (c\lambda', \frac{c}{2}\ln(\frac{1+\sin\varphi'}{1-\sin\varphi'}))$$

$$= (c\arctan(\cot\varphi\cos(\lambda-\lambda_0)), \frac{c}{2}\ln(\frac{1+\cos\varphi\sin(\lambda-\lambda_0)}{1-\cos\varphi\sin(\lambda-\lambda_0)}))$$

Since rotation preserves distances, the distortion under Transversal Mercator projection comes from the Mercator projection. Hence, the measure is given as

$$m(\lambda,\varphi) = \frac{c}{\cos\varphi'} = \frac{c}{\sqrt{1-\sin^2\varphi'}} = \frac{c}{\sqrt{1-\cos^2\varphi\sin^2(\lambda-\lambda_0)}}$$

The value of $c$ for the Universal Transversal Mercatorprojection (UTM) is: $c = 0,9996$.

## 2.2 The ellipsoid

When we replace the sphere by an ellipsoid, things get more complicated. Our method of rotating and then using the normal Mercator does not work for an ellipsoid, since the curves defined by meridians are no longer circles, but ellipses. There are hence many ways of defining the Transversal Mercator projection from the ellipsoid [2] p.159. One solution is discussed in detail in [1], where a formula for a Transversal Mercator projection from the ellipsoid is given as the first terms of a Taylor series:

$$u = N\cos(\varphi)l + \frac{1}{6}N\cos^3(\varphi)(1-\tan^2(\varphi)+\eta^2)l^3 + \frac{1}{120}N\cos^5(\varphi)(5-18\tan^2(\varphi)+\tan^4(\varphi))l^5 + \cdots$$

$$v = B + \frac{1}{2}N\cos^2(\varphi)\tan(\varphi)l^2 + \frac{1}{24}N\cos^4(\varphi)\tan(\varphi)(5-\tan^2(\varphi)+9\eta^2)l^4 + \cdots$$

Here $\eta^2 = \frac{a^2-b^2}{b^2}\cos^2\varphi$, $l = \lambda - \lambda_0$ and $B$ is the distance from the Equator to the parallel $\varphi$ measured along a meridian.

The reader should remind him/herself of the meaning of geographical coordinates on the Ellipsoid.

When calculating corrections to lengths or angles, one may choose the sphere approximating the ellipsoid best in the area of interest. This is the sphere with radius $\sqrt{M_0 N_0}$, where $M_0$ and $N_0$ are the quantities $M$ and $N$ from the first fundamental form evaluated at a chosen point in the area considered. Then the calculations proceed as if the earth was a sphere.

# References


[1] K. Borre, *Landmåling*, Institut for Samfundsudvikling og Planlægning, 1990.

[2] J.R.Snyder, *Flattening the earth*, University of Chicago Press, 1993.

[3] J. McCleary, *Geometry from a Differentiable Viewpoint*, Cambridge University Press, 1994.