Chapter 2

Initial value problems

Our main task in this section will be to prove the basic existence and uniqueness result for ordinary differential equations. The key ingredient will be the contraction principle (Banach fixed point theorem), which we will derive first.

2.1. Fixed point theorems

Let X be a real **vector space**. A **norm** on X is a map $\|.\| : X \to [0, \infty)$ satisfying the following requirements:

(i) $\|0\| = 0$, $\|x\| > 0$ for $x \in X \backslash \{0\}$.

(ii) $\|\lambda x\| = |\lambda| \, \|x\|$ for $\lambda \in \mathbb{R}$ and $x \in X$.

(iii) $\|x + y\| \le \|x\| + \|y\|$ for $x, y \in X$ (**triangle inequality**).

The pair $(X, \|.\|)$ is called a **normed vector space**. Given a normed vector space X, we have the concept of convergence and of a **Cauchy sequence** in this space. The normed vector space is called **complete** if every Cauchy sequence converges. A complete normed vector space is called a **Banach space**.

Clearly $\mathbb{R}^n$ (or $\mathbb{C}^n$) is a Banach space. We will be mainly interested in the following example: Let I be a compact interval and consider the continuous functions $C(I)$ on this interval. They form a vector space if all operations are defined pointwise. Moreover, $C(I)$ becomes a normed space if we define

$$\|x\| = \sup_{t \in I} |x(t)|. \tag{2.1}$$

I leave it as an exercise to check the three requirements from above. Now what about convergence in this space? A sequence of functions $x_n(t)$ converges to x if and only if

$$\lim_{n\to\infty} \|x - x_n\| = \lim_{n\to\infty} \sup_{t\in I} |x_n(t) - x(t)| = 0. \tag{2.2}$$

That is, in the language of real analysis, x_n converges uniformly to x. Now let us look at the case where x_n is only a Cauchy sequence. Then $x_n(t)$ is clearly a Cauchy sequence of real numbers for any fixed $t \in I$. In particular, by completeness of $\mathbb{R}$, there is a limit $x(t)$ for each t. Thus we get a limiting function $x(t)$. Moreover, letting $m \to \infty$ in

$$|x_n(t) - x_m(t)| \le \varepsilon \qquad \forall n, m > N_\varepsilon,\ t \in I \tag{2.3}$$

we see

$$|x_n(t) - x(t)| \le \varepsilon \qquad \forall n > N_\varepsilon,\ t \in I, \tag{2.4}$$

that is, $x_n(t)$ converges uniformly to $x(t)$. However, up to this point we don't know whether $x(t)$ is in our vector space $C(I)$ or not, that is, whether it is continuous or not. Fortunately, there is a well-known result from real analysis which tells us that the uniform limit of continuous functions is again continuous. Hence $x(t) \in C(I)$ and thus every Cauchy sequence in $C(I)$ converges. Or, in other words, $C(I)$ is a Banach space.

You will certainly ask how all these considerations should help us with our investigation of differential equations? Well, you will see in the next section that it will allow us to give an easy and transparent proof of our basic existence and uniqueness theorem based on the following result.

A **fixed point** of a mapping $K : C \subseteq X \to C$ is an element $x \in C$ such that $K(x) = x$. Moreover, K is called a contraction if there is a contraction constant $\theta \in [0, 1)$ such that

$$\|K(x) - K(y)\| \le \theta \|x - y\|, \qquad x, y \in C. \tag{2.5}$$

We also recall the notation $K^n(x) = K(K^{n-1}(x))$, $K^0(x) = x$.

Theorem 2.1 (Contraction principle)**.** *Let C be a (nonempty) closed subset of a Banach space X and let $K : C \to C$ be a contraction, then K has a unique fixed point $\overline{x} \in C$ such that*

$$\|K^n(x) - \overline{x}\| \le \frac{\theta^n}{1-\theta} \|K(x) - x\|, \qquad x \in C. \tag{2.6}$$

Proof. If $x = K(x)$ and $\tilde{x} = K(\tilde{x})$, then $\|x - \tilde{x}\| = \|K(x) - K(\tilde{x})\| \le \theta \|x - \tilde{x}\|$ shows that there can be at most one fixed point.

Concerning existence, fix $x_0 \in C$ and consider the sequence $x_n = K^n(x_0)$. We have

$$\|x_{n+1} - x_n\| \le \theta \|x_n - x_{n-1}\| \le \cdots \le \theta^n \|x_1 - x_0\| \tag{2.7}$$

and hence by the triangle inequality (for $n > m$)

$$\begin{aligned}\|x_n - x_m\| &\leq \sum_{j=m+1}^{n} \|x_j - x_{j-1}\| \leq \theta^m \sum_{j=0}^{n-m-1} \theta^j \|x_1 - x_0\| \\ &\leq \frac{\theta^m}{1-\theta}\|x_1 - x_0\|. \end{aligned} \tag{2.8}$$

Thus x_n is Cauchy and tends to a limit $\overline{x}$. Moreover,

$$\|K(\overline{x}) - \overline{x}\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0 \tag{2.9}$$

shows that $\overline{x}$ is a fixed point and the estimate (2.6) follows after taking the limit $n \to \infty$ in (2.8). □

Question: Why is closedness of C important?

Problem 2.1. *Show that the space $C(I, \mathbb{R}^n)$ together with the sup norm (2.1) is a Banach space.*

Problem 2.2. *Derive Newton's method for finding the zeros of a function $f(x)$,*

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

from the contraction principle. What is the advantage/disadvantage of using

$$x_{n+1} = x_n - \theta\frac{f(x_n)}{f'(x_n)}, \quad \theta > 0,$$

instead?

2.2. The basic existence and uniqueness result

Now we want to use the preparations of the previous section to show existence and uniqueness of solutions for the following **initial value problem** (IVP)

$$\dot{x} = f(t, x), \quad x(t_0) = x_0. \tag{2.10}$$

We suppose $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$.

First of all note that integrating both sides with respect to t shows that (2.10) is equivalent to the following **integral equation**

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{2.11}$$

At first sight this does not seem to help much. However, note that $x_0(t) = x_0$ is an approximating solution at least for small t. Plugging $x_0(t)$ into our integral equation we get another approximating solution

$$x_1(t) = x_0 + \int_{t_0}^{t} f(s, x_0(s))\, ds. \tag{2.12}$$

Iterating this procedure we get a sequence of approximating solutions

$$x_n(t) = K^n(x_0)(t), \qquad K(x)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds. \tag{2.13}$$

Now this observation begs us to apply the contraction principle from the previous section to the fixed point equation $x = K(x)$, which is precisely our integral equation (2.11).

We will set $t_0 = 0$ for notational simplicity and consider only the case $t \geq 0$ to avoid excessive numbers of absolute values in the following estimates.

First of all we will need a Banach space. The obvious choice is $X = C(I)$, where $I = [0, T]$ is some suitable interval containing $t_0 = 0$. Furthermore, we need a closed subset $C \subseteq X$ such that $K : C \to C$. We will try a closed ball of radius δ around x_0, where $\delta > 0$ has to be determined.

Choose $V = [0, T] \times B_\delta(x_0)$, where $B_\delta(x_0) = \{x \in \mathbb{R}^n | \, |x - x_0| \leq \delta\}$. Then

$$|K(x)(t) - x_0| \leq \int_0^t |f(s, x(s))| ds \leq t \max_{(t,x) \in V} |f(t, x)| \tag{2.14}$$

(here the maximum exists by continuity of f and compactness of V) whenever the graph of x lays within V, that is, $\{(t, x(t)) | t \in [0, T]\} \subset V$. Hence, for $t \leq T_0$, where

$$T_0 = \min(T, \frac{\delta}{M}), \qquad M = \max_{(t,x) \in V} |f(t, x)|, \tag{2.15}$$

we have $T_0 \, M \leq \delta$ and the graph of $K(x)$ is again in V.

So if we choose $X = C([0, T_0])$ as our Banach space, with norm $\|x\| = \max_{0 \leq t \leq T_0} |x(t)|$, and $C = \{x \in X \,|\, \|x - x_0\| \leq \delta\}$ as our closed set, then $K : C \to C$ and it remains to show that K is a contraction.

To show this, we need to estimate

$$|K(x)(t) - K(y)(t)| \leq \int_{t_0}^{t} |f(s, x(s)) - f(s, y(s))| ds. \tag{2.16}$$

Clearly, since f is continuous, we know that $|f(s, x(s)) - f(s, y(s))|$ is small if $|x(s) - y(s)|$ is. However, this is not good enough to estimate the integral above. For this we need the following stronger condition. Suppose f is locally **Lipschitz continuous** in the second argument. That is, for every compact set $V \subset U$ the following number

$$L = \sup_{(t,x) \neq (t,y) \in V} \frac{|f(t, x) - f(t, y)|}{|x - y|} \tag{2.17}$$

(which depends on V) is finite. Then,

$$\begin{aligned}\int_0^t |f(s,x(s)) - f(s,y(s))|ds &\leq L\int_0^t |x(s)-y(s)|ds\\ &\leq L\,t \sup_{0\leq s\leq t} |x(s)-y(s)| \qquad (2.18)\end{aligned}$$

provided the graphs of both $x(t)$ and $y(t)$ lie in V. In other words,

$$\|K(x) - K(y)\| \leq L\,T_0 \|x-y\|, \quad x \in C. \qquad (2.19)$$

Moreover, choosing $T_0 < L^{-1}$ we see that K is a contraction and existence of a unique solution follows from the contraction principle:

Theorem 2.2 (Picard-Lindelöf)**.** *Suppose $f \in C(U,\mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0,x_0) \in U$. If f is locally Lipschitz continuous in the second argument, then there exists a unique local solution $\overline{x}(t)$ of the IVP (2.10).*

The procedure to find the solution is called **Picard iteration**. Unfortunately, it is not suitable for actually finding the solution since computing the integrals in each iteration step will not be possible in general. Even for numerical computations it is of no great help, since evaluating the integrals is too time consuming. However, at least we know that there is a unique solution to the initial value problem.

In many cases, f will be even differentiable. In particular, note that $f \in C^1(U,\mathbb{R}^n)$ implies that f is locally Lipschitz continuous (see the problems below).

Lemma 2.3. *Suppose $f \in C^k(U,\mathbb{R}^n)$, $k \geq 1$, where U is an open subset of $\mathbb{R}^{n+1}$, and $(t_0,x_0) \in U$. Then the local solution $\overline{x}$ of the IVP (2.10) is C^{k+1}.*

Proof. Let $k = 1$. Then $\overline{x}(t) \in C^1$ by the above theorem. Moreover, using $\dot{\overline{x}}(t) = f(t,\overline{x}(t)) \in C^1$ we infer $\overline{x}(t) \in C^2$. The rest follows from induction. □

Problem 2.3. *Show that $f \in C^1(\mathbb{R})$ is locally Lipschitz continuous. In fact, show that*

$$|f(y) - f(x)| \leq \sup_{\varepsilon\in[0,1]} |f'(x+\varepsilon(y-x))||x-y|.$$

Generalize this result to $f \in C^1(\mathbb{R}^m,\mathbb{R}^n)$.

Problem 2.4. *Are the following functions Lipschitz continuous at 0? If yes, find a Lipschitz constant for some interval containing 0.*

(i) $f(x) = \frac{1}{1-x^2}$.

(ii) $f(x) = |x|^{1/2}$.

(iii) $f(x) = x^2 \sin(\frac{1}{x})$.

Problem 2.5. *Apply the Picard iteration to the first-order linear equation*

$$\dot{x} = x, \qquad x(0) = 1.$$

Problem 2.6. *Investigate uniqueness of the differential equation*

$$\dot{x} = \begin{cases} -t\sqrt{|x|}, & x \ge 0 \\ t\sqrt{|x|}, & x \le 0 \end{cases}.$$

2.3. Some extensions

In this section we want to derive some further extensions of the Picard-Lindelöf theorem. They are of a more technical nature and can be skipped on first reading.

As a preparation we need a slight generalization of the contraction principle. In fact, looking at its proof, observe that we can replace θ^n by any other summable sequence θ_n (Problem 2.8).

Theorem 2.4 (Weissinger)**.** *Let C be a (nonempty) closed subset of a Banach space X. Suppose $K : C \to C$ satisfies*

$$\|K^n(x) - K^n(y)\| \le \theta_n \|x - y\|, \qquad x, y \in C, \tag{2.20}$$

with $\sum_{n=1}^{\infty} \theta_n < \infty$. Then K has a unique fixed point $\overline{x}$ such that

$$\|K^n(x) - \overline{x}\| \le \left(\sum_{j=n}^{\infty} \theta_j \right) \|K(x) - x\|, \qquad x \in C. \tag{2.21}$$

Our first objective is to give some concrete values for the existence time T_0. Using Weissinger's theorem instead of the contraction principle, we can avoid the restriction $T_0 < L^{-1}$:

Theorem 2.5 (Picard-Lindelöf)**.** *Suppose $f \in C(U, \mathbb{R}^n)$, where U is an open subset of $\mathbb{R}^{n+1}$, and f is locally Lipschitz continuous in the second argument. Choose $(t_0, x_0) \in U$ and $\delta > 0$, $T > t_0$ such that $[t_0, T] \times B_\delta(x_0) \subset U$. Set*

$$M(t) = \int_{t_0}^{t} \sup_{x \in B_\delta(x_0)} |f(s, x)| ds, \tag{2.22}$$

$$L(t) = \sup_{x \neq y \in B_\delta(x_0)} \frac{|f(t, x) - f(t, y)|}{|x - y|}. \tag{2.23}$$

Note that $M(t)$ is nondecreasing and define T_0 via

$$M(T_0) = \delta. \tag{2.24}$$

Then the unique local solution $\overline{x}(t)$ of the IVP (2.10) is given by

$$\overline{x} = \lim_{n \to \infty} K^n(x_0) \in C^1([t_0, T_0], B_\delta(x_0)) \tag{2.25}$$