Fra Horia Corneans Noter til analysekurserne.

Theorem 6.3. *Consider the initial value problem:*

$$\mathbf{y}'(t) = \mathbf{f}(t, \mathbf{y}(t)), \quad \mathbf{y}(t_0) = \mathbf{y}_0. \tag{6.7}$$

Define $\delta_1 := \min\{\delta_0, r_0/M, 1/L\}$. *Then there exists a solution* $\mathbf{y} : (t_0 - \delta_1, t_0 + \delta_1) \mapsto \overline{B_{r_0}(\mathbf{y}_0)}$, *which is unique.*

Proof. Take some $0 < \delta < \delta_1$ and define the compact interval $K := [t_0 - \delta, t_0 + \delta] \subset \mathbb{R}$. Then any continuous function $\phi : K \to \mathbb{R}^d$ is automatically bounded, and since the Euclidean space $Y = \mathbb{R}^d$ is a Banach space, we can conclude from Proposition 6.2 that the space $(C(K; \mathbb{R}^d), d_\infty)$ of continuous functions defined on the compact K with values in $\mathbb{R}^d$ is a complete metric space.

Define

$$X := \{\mathbf{g} \in C(K; \mathbb{R}^d) : \quad \mathbf{g}(t) \in \overline{B_{r_0}(\mathbf{y}_0)}, \ \forall t \in K\}. \tag{6.8}$$

Lemma 6.4. *The metric space* (X, d_∞) *is complete.*

Proof. Consider a Cauchy sequence $\{\mathbf{g}_n\}_{n\geq 1} \subset X$. Because $(C(K; \mathbb{R}^d), d_\infty)$ is complete, we can find $\mathbf{g} \in C(K; \mathbb{R}^d)$ such that $\lim_{n\to\infty} d_\infty(\mathbf{g}_n, \mathbf{g}) = 0$. Thus for every $t \in K$ we have

$$\mathbf{g}(t) = \lim_{n\to\infty} \mathbf{g}_n(t), \quad \lim_{n\to\infty} \|\mathbf{g}_n(t) - \mathbf{g}(t)\| = 0.$$

Since by assumption $\|\mathbf{g}_n(t) - \mathbf{y}_0\| \leq r_0$ for all t and n, we have

$$\|\mathbf{g}(t) - \mathbf{y}_0\| = \lim_{n\to\infty} \|\mathbf{g}_n(t) - \mathbf{y}_0\| \leq r_0, \quad \forall t \in K,$$

which implies that $\mathbf{g} \in X$. $\square$

Lemma 6.5. *Define the map* $F : X \to C(K; \mathbb{R}^d)$

$$[F(\mathbf{g})](t) := \mathbf{y}_0 + \int_{t_0}^{t} \mathbf{f}(s, \mathbf{g}(s)) ds, \quad \forall t \in K,$$

where $\mathbf{f}$ *obeys (6.5). Then (i) the range of* F *belongs to* X *and (ii)* $F : X \to X$ *is a contraction.*

Proof.

(i). Since f_j are continuous real valued functions, we have that

$$K \ni s \mapsto f_j(s, \mathbf{g}(s)) \in \mathbb{R}$$

are also continuous, thus Riemann integrable. Because $\mathbf{g}(s) \in \overline{B_{r_0}(\mathbf{y}_0)}$ for all $s \in K$, we have that $(s, \mathbf{g}(s)) \in H_0$. The integral from the definition of F defines a vector $\mathbf{u}(t)$ with components

$$u_j(t) := \int_{t_0}^{t} f_j(s, \mathbf{g}(s)) ds, \quad 1 \leq j \leq d.$$

Denote by $t_1 := \min\{t_0, t\}$ and $t_2 := \max\{t_0, t\}$. Then we have:

$$||\mathbf{u}(t)||^2 = \sum_{j=1}^{d} u_j^2(t) = \int_{t_0}^{t} \left(\sum_{j=1}^{d} u_j(t) f_j(s, \mathbf{g}(s)) \right) ds \leq \int_{t_1}^{t_2} ||\mathbf{u}(t)|| \, ||\mathbf{f}(s, \mathbf{g}(s))|| ds$$

where in the last inequality we used the Cauchy-Schwarz inequality. Hence we may write:

$$\left\| \int_{t_0}^{t} \mathbf{f}(s, \mathbf{g}(s)) ds \right\| \leq \int_{t_1}^{t_2} ||\mathbf{f}(s, \mathbf{g}(s))|| ds.$$

From (6.6) we have $\sup_{s\in K} \|\mathbf{f}(s, \mathbf{g}(s))\| \leq M$, hence:

$$\|[F(\mathbf{g})](t) - \mathbf{y}_0\| = \|\mathbf{u}(t)\| \leq \int_{t_1}^{t_2} ||\mathbf{f}(s, \mathbf{g}(s))|| ds \leq M\delta < r_0, \quad \forall t \in K,$$

Implicit funktions sætningen.

We can now formulate the implicit function theorem.

Theorem 7.4. *Let $U \subset \mathbb{R}^d$ be an open set and $\mathbf{h} : U \mapsto \mathbb{R}^m$ be a $C^1(U;\mathbb{R}^m)$ function. Assume that there exists a point $\mathbf{a} = [\mathbf{u_a}, \mathbf{w_a}] \in U$ such that $\mathbf{h}(\mathbf{a}) = 0$ and the $m \times m$ partial Jacobi matrix $[D_\mathbf{u}\mathbf{h}(\mathbf{a})]$ is invertible. Then there exists an open set $E \subset \mathbb{R}^n$ containing $\mathbf{w_a}$ and a map $\mathbf{f} : E \mapsto \mathbb{R}^m$ which obeys $\mathbf{f}(\mathbf{w_a}) = \mathbf{u_a}$ and $\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}]) = 0$ for all $\mathbf{w} \in E$. Moreover, the matrix $[D_\mathbf{u}\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}])]$ is invertible if $\mathbf{w} \in E$ and all entries of its inverse are continuous on E. Finally, $\mathbf{f}$ is continuously differentiable on E and we have:*

$$[D\mathbf{f}(\mathbf{w})] = -[D_\mathbf{u}\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}])]^{-1}\,[D_\mathbf{w}\mathbf{h}([\mathbf{f}(\mathbf{w}), \mathbf{w}])] \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^m), \quad \forall \mathbf{w} \in E. \tag{7.4}$$

Moreover, by possibly shrinking E, we obtain: There is $V \subset U$ open s.t. $\underline{a} \in V$ and if $x \in V$ satisfies $h(\underline{x}) = \underline{0}$, then there is a $w_x \in E$ s.t.

$$\underline{x} = (f(w_x), w_x)$$

(i.e. $h^{-1}(\underline{0}) \cap E = \{ (f(w), w) \mid w \in E \}$

Furthermore: If h is C^∞, so is f.

Here is the Inverse Function Theorem:

Theorem 8.3. *Let $\mathcal{O} \subset \mathbb{R}^m$ be an open set containing $\mathbf{u}_0$. Let $\mathbf{g} \in C^1(\mathcal{O}; \mathbb{R}^m)$ such that $[D\mathbf{g}(\mathbf{u}_0)] \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ is invertible, and $\mathbf{g}$ is injective on $\mathcal{O}$. Then there exists an open ball $E \subset \mathbb{R}^m$ which contains $\mathbf{w}_0 := \mathbf{g}(\mathbf{u}_0)$, and a function $\mathbf{f} : E \mapsto \mathcal{O}$ such that the following facts hold true:*

(i). The set $V = \mathbf{f}(E)$ equals $\mathbf{g}^{-1}(E)$ and is open in $\mathbb{R}^m$;
(ii). $\mathbf{g}(\mathbf{f}(\mathbf{w})) = \mathbf{w}$ on E and $\mathbf{f}(\mathbf{g}(\mathbf{u})) = \mathbf{u}$ on V, hence they are local inverses to each other;
(iii). The function $\mathbf{f}$ is a $C^1(V)$ function, $[D\mathbf{g}(\mathbf{f}(\mathbf{w}))]$ is invertible on E and we have:

$$[D\mathbf{f}(\mathbf{w})] = [D\mathbf{g}(\mathbf{f}(\mathbf{w}))]^{-1}.$$

Furthermore: If g is C^∞, then g^{-1} is C^∞

9 Brouwer's fixed point theorem

We say that $K \subset \mathbb{R}^d$ is convex if for every $\mathbf{x}, \mathbf{y} \in K$ we have that $(1-t)\mathbf{x} + t\mathbf{y} \in K$ for all $0 \le t \le 1$. A set K is called a convex body if K is convex, compact, and with at least one interior point.

Theorem 9.1. *Let $K \subset \mathbb{R}^d$ be a convex body. Let $\mathbf{f} : K \mapsto K$ be a continuous function which invariates K. Then $\mathbf{f}$ has a (not necessarily unique) fixed point, that is a point $\mathbf{x} \in K$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{x}$.*

Proof. The first thing we do is to reduce the problem from a general convex body to the unit ball in $\mathbb{R}^d$. We will show that there exists a bijection $\varphi : K \mapsto \overline{B_1(0)}$, which is continuous and with continuous inverse (a homeomorphism). If this is true, then it is enough to show that the function $\varphi \circ \mathbf{f} \circ \varphi^{-1} : \overline{B_1(0)} \mapsto \overline{B_1(0)}$ has a fixed point $\mathbf{a} \in \overline{B_1(0)}$. In that case, $\mathbf{x} = \varphi^{-1}(\mathbf{a}) \in K$.

Lemma 9.2. *Any convex body in $\mathbb{R}^d$ is homeomorphic with the closed unit ball $\overline{B_1(0)}$.*

Proof. Let $\mathbf{x}_0$ be an interior point of K. There exists $r > 0$ such that $\overline{B_r(\mathbf{x}_0)} \subset K$. Define the continuous map $g : K \mapsto \mathbb{R}^d$ given by $g(\mathbf{x}) := (\mathbf{x} - \mathbf{x}_0)r^{-1}$. Define $\tilde{K} := g(K)$. It is easy to see that $\tilde{K}$ is a convex and compact set. Moreover, the function $g : K \mapsto \tilde{K}$ is invertible and $g^{-1}(\mathbf{y}) = r\mathbf{y} + \mathbf{x}_0$. Both g and g^{-1} are continuous, and for every $\mathbf{y} \in \mathbb{R}^d$ with $||\mathbf{y}|| \le 1$ we have that $r\mathbf{y} + \mathbf{x}_0 \in \overline{B_r(\mathbf{x}_0)} \subset K$, thus $\mathbf{y} \in \tilde{K}$. This shows that $\overline{B_1(0)} \subset \tilde{K}$, thus $\tilde{K}$ is a convex body containing the closed unit ball.