Differential Geometry. Activity 6.

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Aalborg 2016

Surface patch. Atlas for a regular surface

Definition

Let $S \subset \mathbf{R}^3$.

- A surface patch for S is a map $\sigma : U \to \mathbf{R}^3$ on an open set $U \subseteq \mathbf{R}^2$ so that there exists an open set $V \subseteq \mathbf{R}^3$ with $\sigma(U) = S \cap V$ and such that
  1. σ is smooth;
  2. σ is regular, i.e., $D_q\sigma : \mathbf{R}^2 \to \mathbf{R}^3$ has (maximal) rank 2 for all $q \in U$;
  3. The "restriction" $\sigma : U \to S \cap V$ is a homeomorphism, i.e., it has a continuous inverse map.

  Properties of $\sigma: U \to \mathbb{R}^3$ Checking does not involve S

- An atlas for S is a collection of surface patches for S such that every point $p \in S$ is contained in the image of at least one patch in that collection.
- If S has an atlas, it is called a regular surface.

Eksempler:

Grafen for en glat funktion:

$$f: U \to \mathbb{R} \quad U \subset \mathbb{R}^2, \text{åben}$$

$$S_1 = \{(u,v,f(u,v)) \mid (u,v) \in U\}$$

$$S_2 = \{(u,f(u,v),v) \mid (u,v) \in U\}$$

$$S_3 = \{(f(u,v),u,v) \mid (u,v) \in U\}$$

er alle regulære flader.

Atlas: ét kort

$\sigma: U \to \mathbb{R}^3 \quad \sigma(u,v) = (u,f(u,v),v)$ etc.

Hvorfor er σ^{-1} kontinuert?

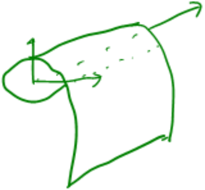
- $\pi(x,y,z) = (x,z)$ er kontinuert (endda lineær)
- $\sigma^{-1} = \pi|_{S_2}$

Eksempler, der <u>ikke</u> er regulære flader:

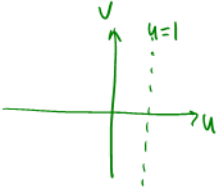
$$\sigma(u,v) = (u^2, u\cdot(u^2-1), v)$$

$$U = \{(u,v) \in \mathbb{R}^2 \mid u < 1\}$$

$$S = \sigma(U)$$



$\sigma : U \to S \cap \mathbb{R}^3$ bijektiv



MEN σ^{-1} er <u>ikke</u> kontinuert

Level surfaces

Question. When is the set of solutions of an equation $f(x, y, z) = c$ a regular surface?

Theorem

Let $S \subset \mathbf{R}^3$ have the property:
For every $\mathbf{p} = (x_0, y_0, z_0) \in S$ there is an open subset $p \in W \subseteq \mathbf{R}^3$, a smooth function $f : W \to \mathbf{R}$, $c \in \mathbf{R}$, such that

1. $S \cap W = \{(x, y, z) \in W | \ f(x, y, z) = c\}$;
2. $\nabla f(x, y, z) \neq \mathbf{0}$ *for all* $(x, y, z) \in S \cap W$.

Then S is a regular surface.

Proof.

Apply the Implicit Function Theorem to produce a graph patch in a neighbourhood of p. □

Korollar / specialtilfælde

Lad $f: W \to \mathbb{R}$, $W \subseteq \mathbb{R}^3$, f glat.

Lad $S = \{(x,y,z) \in W \mid f(x,y,z) = a\}$

Hvis $\nabla f(p) \neq \underline{0}$ for alle $p \in S$, så er S en regulær flade.