Differential Geometry. Activity 9.

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Aalborg 2016

Surface patch. Atlas for a regular surface

Definition

Let $S \subset \mathbf{R}^3$.

- A surface patch for S is a map $\sigma : U \to \mathbf{R}^3$ on an open set $U \subseteq \mathbf{R}^2$ so that there exists an open set $V \subseteq \mathbf{R}^3$ with $\sigma(U) = S \cap V$ and such that
  1. σ is smooth;
  2. σ is regular, i.e., $D_q\sigma : \mathbf{R}^2 \to \mathbf{R}^3$ has (maximal) rank 2 for all $q \in U$;
  3. The restriction $\sigma : U \to S \cap V$ is a homeomorphism, i.e., it has a continuous inverse map.

- An atlas for S is a collection of surface patches for S such that every point $p \in S$ is contained in the image of at least one patch in that collection.
- If S has an atlas, it is called a regular surface.

Atlas

Level surfaces

Question. When is the set of solutions of an equation $f(x, y, z) = c$ a regular surface?

Theorem

Let $S \subset \mathbf{R}^3$ have the property:
For every $\mathbf{p} = (x_0, y_0, z_0) \in S$ there is an open subset $p \in W \subseteq \mathbf{R}^3$, a smooth function $f : W \to \mathbf{R}$, $c \in \mathbf{R}$, such that

1. $S \cap W = \{(x, y, z) \in W \mid f(x, y, z) = c\}$;
2. $\nabla f(x, y, z) \neq \mathbf{0}$ *for all* $(x, y, z) \in S \cap W$.

Then S is a regular surface.

Proof.

Apply the Implicit Function Theorem to produce a graph patch in a neighbourhood of p. □

Parametrizations and reparametrizations

Theorem

Given a patch $\sigma : U \to S \subset \mathbf{R}^3$ for the smooth surface S and a point $p \in U$.

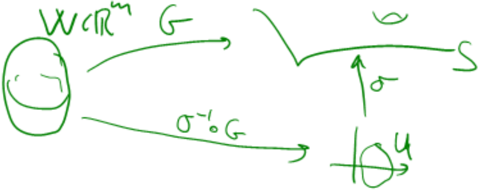
Composition with projection *There exists an open subset $U' \subset U$ containing p and a projection $\pi : \mathbf{R}^3 \to \mathbf{R}^2$ to one of the coordinate planes such that the composition $\pi \circ \sigma : U' \to V' = (\pi \circ \sigma)(U')$ is a diffeomorphism.*

Local smooth "inverse" *With U' as above, the restriction $\sigma|_{U'}$ has a smooth "inverse", i.e., there exists a smooth map $F : V \to U'$, $\sigma(U') \subset V \subset \mathbf{R}^3$, V open, such that $(F \circ \sigma)(u, v) = (u, v)$.*

Theorem

Local graph patch *With U' and V' as above, there exists a smooth reparametrization diffeomorphism $\Phi : V' \to U'$ such that* $(\sigma \circ \Phi)(u, v) = (u, v, f(u, v))$ *with $f : V' \subset \mathbf{R}^3$ smooth (up to order).*

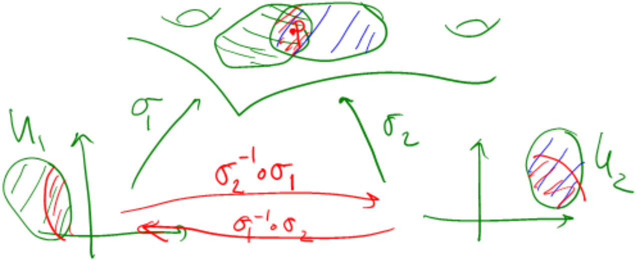
Smooth maps to S *A map $G : W \to S$, $W \subset \mathbb{R}^m$ is smooth at $r \in W$ if and only if $\sigma^{-1} \circ G$ is smooth at r for a patch σ around $G(r)$ on S*



Transition functions

Given two regular coordinate patches $\sigma_i : U_i \to V_i \cap S$. They define a transition function $\sigma_2^{-1} \circ \sigma_1 : \sigma_1^{-1}(V_1 \cap V_2) \to \sigma_2^{-1}(V_1 \cap V_2)$
– a diffeomorphism between plane open sets.

Interpretation: Change of coordinates is smooth (both ways)!



Smooth maps between surfaces

Given smooth surfaces S_1 and S_2 and $p \in S_1$.

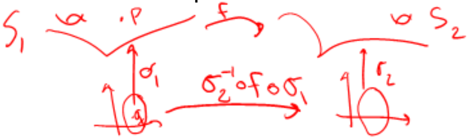
Definition

A map $f : S_1 \to S_2$ is called smooth at p, if there are surface patches $\sigma_i : U_i \to S_i$ and $q \in U_1$ such that $\sigma_1(q) = p$ and $f(\sigma_1(U_1)) \subset \sigma_2(U_2)$ and such that the composite map

$$g = \sigma_2^{-1} \circ f \circ \sigma_1 : U_1 \to U_2$$

is smooth at q.
f is called smooth if it is smooth at all $p \in S_1$.

This definition is independent of the choice of patches: Transition functions!

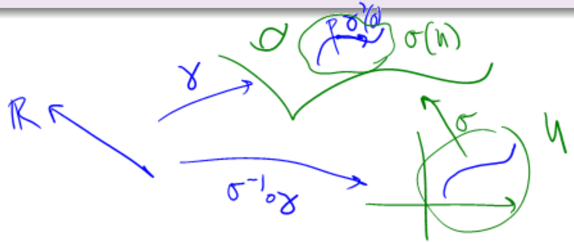


Tangent planes

Smooth curves on a smooth surface S

Definition

- A smooth space curve with parametrization $\gamma : I \to \mathbf{R}^3$ is a curve on S if $\gamma(t) \in S$ for all $t \in S$.
- The tangent plane $T_{\mathbf{p}}S$ consists of the velocity vectors $\dot{\gamma}(0)$ for all curves γ on S with $\gamma(0) = \mathbf{p}$.



Theorem

Let $\sigma : U \to V \cap S$ denote a coordinate patch with $\sigma(\mathbf{q}) = \mathbf{p}$.

- *For a smooth curve $\gamma : I \to S \subset \mathbf{R}^3$ with $\gamma(t_0) = \mathbf{p}$ there exists an interval $J \subset I, t_0 \in J$, and a smooth curve $\delta : J \to U$ such that $\gamma = \sigma \circ \delta$.* $\delta(t) = \sigma^{-1} \circ \gamma(t)$
  (All curves come locally from a smooth curve on a patch.)
- *$D_{\mathbf{q}}\sigma : \mathbf{R}^2 \to T_{\mathbf{p}}S$ is a linear isomorphism.*
- *$T_{\mathbf{p}}S$ is a 2-dimensional linear subspace of $\mathbf{R}^3$.*

$D_q\sigma : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$D_q\sigma(\mathbb{R}^2) = T_pS$

$D_q\sigma : \mathbb{R}^2 \to T_pS$

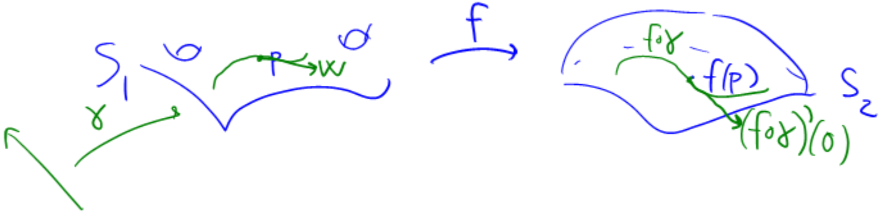
$(D_q\sigma)^{-1} : T_pS \to \mathbb{R}^2$

$w = \gamma'(0)$ $(D_q\sigma)^{-1}(w) = (\sigma^{-1} \circ \gamma)'(0)$

The differential of a smooth map

Definition (Differential $D_pf : T_pS_1 \to T_{f(p)}S_2$ of a smooth map f at $p \in S$)

Represent a tangent vector $\mathbf{w} = \dot{\gamma}(0)$, γ a smooth curve on S_1, $\gamma(0) = p$.
$D_p(f)(\mathbf{w}) = (f \circ \gamma)\dot{}(0)$ – the tangent to the image curve $(f \circ \gamma)$ at $f(p)$.



Theorem

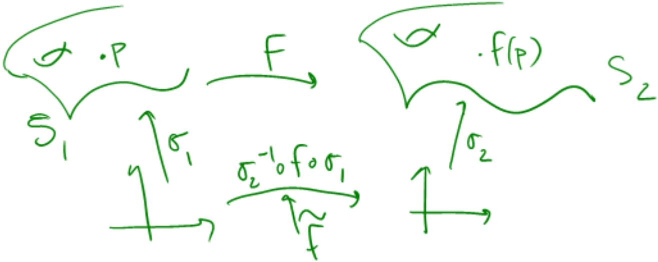
- *The definition of $D_p(f)$ does not depend on the particular choice of the curve γ, only on its tangent $\mathbf{w} = \dot{\gamma}(0)$.*
- *$D_p f : T_p S_1 \to T_{f(p)} S_2$ is linear.*

Proof.

Choose surface patches $\sigma_i : U_i \to S_i$ – as in the definition of a smooth map. Then $\gamma(t) = \sigma_1(\delta(t))$ for some smooth curve δ with $\delta(0) = q$.
$\delta(t) = \sigma_1^{-1} \circ \gamma(t)$
Hence, with $g = \sigma_2^{-1} \circ f \circ \sigma_1$,
$D_p(f)(\mathbf{w}) = (f \circ \gamma)^{\cdot}(0) = (f \circ \sigma_1 \circ \delta)^{\cdot}(0) = (\sigma_2 \circ g \circ \delta)^{\cdot}(0) = D_q(\sigma_2 \circ g)(\dot{\delta}(0))$

- depends only on $\dot{\delta}(0) = (D_q \sigma_1)^{-1}(\dot{\gamma}(0)) = (D_q \sigma_1)^{-1}(\mathbf{w})$.
- $D_p f = D_{g(q)} \sigma_2 \circ D_q g \circ (D_q \sigma_1)^{-1}$ is linear.

□



$$
\begin{tikzcd}
T_p S_1 & & T_{f(p)} S_2 \\
\mathbb{R}^2 \arrow[u, "D_q \sigma_1", "\cong"'] \arrow[rr, bend left, "D_q \tilde{f}"'] & & \mathbb{R}^2 \arrow[u, "D_q \sigma_2", "\cong"']
\end{tikzcd}
$$

Normal vectors / Orientable Surfaces

<u>Def:</u> S, a regular surface, is orientable, if there is an Atlas on S, such that for σ_1, σ_2 in the atlas

$$\det\left(D_q(\sigma_2^{-1}\circ\sigma_1)\right) > 0 \quad \text{for all } q \in \sigma_1^{-1}(\sigma_2(u_2)\cap\sigma_1(u_1))$$

<u>Equivalently:</u>

$$\frac{\sigma_{1u}\times\sigma_{1v}}{|\sigma_{1u}\times\sigma_{1v}|}(q) = \frac{\sigma_{2u}\times\sigma_{2v}}{|\sigma_{2u}\times\sigma_{2v}|}(q)$$