Repetition. Kurver.

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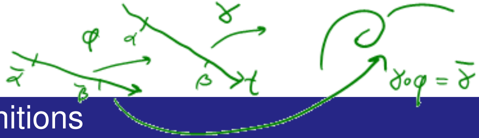


(Re)-parametrizations – Definitions

parametrized curve: $\gamma : I \to \mathbf{R}^n$ C^∞, I an open interval

regular parametrization: $\dot\gamma(t) \neq \mathbf{0}$ for all $t \in I$

unit-speed parametrization: $||\dot\gamma(s)|| = 1$ for all $s \in I$

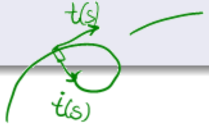
reparametrization: $\bar\gamma = \gamma \circ \varphi$, $\varphi : \bar I \to I$ a bijective diffeomorphism.

Arc length function: $s(t) = \int_{t_0}^{t} ||\dot\gamma(u)||du$

Reparametrization by arc length
$\varphi^{-1}(t) = s(t)$, $\bar\gamma(s) = \gamma(s^{-1}(s))$

(Unit) tangent vectors for regular curves

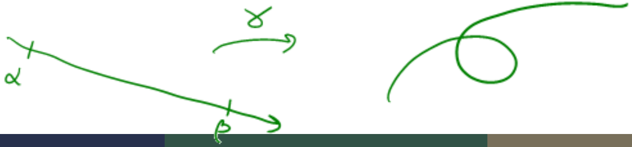
- $\dot\gamma(t) = v(t)\mathbf{t}(t)$ is tangent to the curve at $\gamma(t)$; $v(t)$ is the speed at time t; $\mathbf{t}(t)$ is the unit tangent vector at $\gamma(t)$.
- $\dot{\mathbf{t}}(s) \cdot \mathbf{t}(s) = 0$ for all s for a unit speed curve.



When does the solution of an equation
$f(x, y) = 0$, $f \in C^\infty(U, \mathbf{R})$, $U \subseteq \mathbf{R}^2$ open, describe a regular curve?

Theorem

- *Suppose $f(x_0, y_0) = 0$ and $\nabla f(x_0, y_0) \neq \mathbf{0}$.*
  Then there exists a regular parametrized curve $\gamma(t)$ on an interval $(-\varepsilon, \varepsilon) \to U$ such that $\gamma(0) = (x_0, y_0)$ and such that $f(\gamma(t)) = 0$.
- *Given a regular parametrized plane curve γ on I.*
  For every $t_0 \in I$, there is an open set $U \subset \mathbf{R}^2$ containing $\gamma(t_0)$, a smooth function $f : U \to \mathbf{R}^2$ and $\delta > 0$ such that $f(\gamma(t)) = 0$ for all $t_0 - \delta < t < t_0 + \delta$.



Concepts

$\gamma : I \to \mathbf{R}^n, n = 2, 3,$ a unit-speed parametrization.

Unit tangent $\mathbf{t}(s) = \dot{\gamma}(s)$

(Principal) normal $\mathbf{n}(s) = \begin{cases} \hat{\mathbf{t}}(s) & n = 2 \\ \frac{\dot{\mathbf{t}}(s)}{||\dot{\mathbf{t}}||} & n = 3, \dot{\mathbf{t}}(s) \neq \mathbf{0} \end{cases}$

Curvature Definition: $\dot{\mathbf{t}}(s) = \kappa(s)\mathbf{n}(s)$

Inflection point $\kappa(s) = 0 \Leftrightarrow \dot{\mathbf{t}}(s) = 0$

Binormal $\mathbf{b}(s) = \mathbf{t}(s) \times \mathbf{n}(s)$

Torsion Definition: $\dot{\mathbf{b}}(s) = -\tau(s)\mathbf{n}(s)$

OBS. Only defined when $\kappa(s) \neq 0$ and only in $\mathbb{R}^3$

Need: $\dot{\underline{b}}(s) \parallel \underline{n}(s)$. We gave a proof of that

Frenet-Serret equations

- Frenet matrix $\mathbf{F} = [\mathbf{t}\ \mathbf{n}\ \mathbf{b}] \in SO(3)$ – moving frame (function of s in matrix form)
- Its derivative 3×3-matrix $\dot{\mathbf{F}} = [\dot{\mathbf{t}}\ \dot{\mathbf{n}}\ \dot{\mathbf{b}}]$ – connected to $\mathbf{F}$ by the matrix equation $\dot{\mathbf{F}} = \mathbf{FA}$
- A is a 3×3 skew-symmetric matrix of the form
$$A(s) = \begin{bmatrix} 0 & -\kappa(s) & 0 \\ \kappa(s) & 0 & -\tau(s) \\ 0 & \tau(s) & 0 \end{bmatrix}.$$

A skew + $\dot{t} = \kappa \underline{n}$
$\Rightarrow \dot{b} \parallel n$

- **FS-equations**

$$\begin{aligned} \dot{\mathbf{t}} &= \kappa \mathbf{n} \\ \dot{\mathbf{n}} &= -\kappa \mathbf{t} + \tau \mathbf{b} \\ \dot{\mathbf{b}} &= -\tau \mathbf{n} \end{aligned}$$

Let $\gamma : I \to \mathbf{R}^3$ denote a regular parametrization.Then

Moving frame $\mathbf{t}(t) = \frac{\dot\gamma(t)}{||\dot\gamma(t)||}, \mathbf{b}(t) = \frac{\dot\gamma(t)\times\ddot\gamma(t)}{||\dot\gamma(t)\times\ddot\gamma(t)||}, \mathbf{n}(t) = \mathbf{b}(t)\times\mathbf{t}(t)$.

Curvature $\kappa(t) = \frac{||\dot\gamma(t)\times\ddot\gamma(t)||}{||\dot\gamma(t)||^3}$;

plane: $\kappa(t) = \frac{\det[\dot\gamma(t)\ \ddot\gamma(t)]}{||\dot\gamma(t)||^3}$.

Torsion $\tau(t) = \frac{\det[\dot\gamma(t)\ \ddot\gamma(t)\ \dddot\gamma(t)]}{||\dot\gamma(t)\times\ddot\gamma(t)||^2}$

Proof relies on:

$$\begin{aligned} \dot\gamma &= v\mathbf{t} \\ \ddot\gamma &= \dot v\mathbf{t} + v^2\kappa\mathbf{n} \end{aligned}$$

$v(t) = |\dot\gamma(t)|$

$\dot\gamma\times\ddot\gamma = v^3\kappa\mathbf{b}$
$\mathbf{b}\cdot\ddot\gamma = 0 \Rightarrow v\dot{\mathbf{b}}\cdot\ddot\gamma + \mathbf{b}\cdot\dddot\gamma = 0$
$(\dot\gamma\times\ddot\gamma)\cdot\dddot\gamma = v^3\kappa\mathbf{b}\cdot\dddot\gamma = -v^4\kappa\dot{\mathbf{b}}\cdot\ddot\gamma = v^4\kappa\tau\mathbf{n}\cdot\ddot\gamma = v^6\kappa^2\tau = ||\dot\gamma\times\ddot\gamma||^2\tau.$

Plane

1. Existence: For every smooth function $k : I \to \mathbf{R}$ there exists a unit-speed curve $\gamma : I \to \mathbf{R}^2$ with curvature function $\kappa(s) = k(s)$.
2. "Uniqueness": Two unit-speed curves $\gamma_1, \gamma_2 : I \to \mathbf{R}^2$ have the same curvature functions $\kappa_1(s) = \kappa_2(s)$ if and only if there exists a directed isometry $S \in Iso_+(2)$ such that $\gamma_2 = S \circ \gamma_1$.

Space

1. Existence: For every pair of smooth functions $k : I \to]0, \infty[,\ t : I \to \mathbf{R}$, there exists a unit-speed curve $\gamma : I \to \mathbf{R}^3$ with curvature function $\kappa(s) = k(s)$ and torsion function $\tau(s) = t(s)$.
2. "Uniqueness": Two unit-speed curves $\gamma_1, \gamma_2 : I \to \mathbf{R}^3$ without inflection point have the same curvature functions $\kappa_1(s) = \kappa_2(s)$ and same torsion functions $\tau_1(s) = \tau_2(s)$ if and only if there exists a direct isometry $S \in Iso_+(3)$ such that $\gamma_2 = S \circ \gamma_1$.