Welcome to the Geometry course! This course will consist of activities in which the lecturer participates at different levels; see the course webpage. The course will be held in English if any guest students participate, otherwise we will of course speakDanish. The plans and also the book are in English in any case. This first activity will be of type 2: 2 hours of introductory lectures followed by elementary exercises. Questions to the lecturer at the end of the morning session.

Schedule

Type: 2

8:15 – 10:00 Lecture in G5-109.

10:00 – 11:30 Exercises in group rooms

11:30 – 12:00 Your questions and answers (hopefully) from the lecturer.

This course is special in the sense that you will need to integrate knowledge from *several* previous courses:
Differential Geometry applies tools, primarily from

Calculus and Mathematical Analysis Differentiation, integration, solvability of systems of ordinary differential equations, inverse and implicit function theorems

abstract Linear Algebra quadratic forms, eigenvalues and eigenvectors, spectral theorem

to serve geometric purposes.

Lectures

Content. Aims.

General

To begin with, we will give a short outlook and set this course into perspective. It deals mainly with mathematical tools and methods that let us analyse geometric objects in the three-dimensional world with standard Euclidean geometry as a point of departure. The main objects of study are 1D-curves in plane and space and 2D-surfaces in space. Most of the methods and results were already known to the great mathematician Carl Friedrich Gauss in the first half of the 19th century. Gauss was responsible for the triangulation of the Hannover region and this lead him to questions, theory and results, which are fundamental for our understanding of the world today. At the end of the course, it will be possible to take a glimpse into higher-dimensional and alternative geometries.

Curves and their parametrizations

Parametrizations: A curve is described by a *parametrization*, ie by a *smooth* $= C^\infty$ *vector function* of *one* variable. We will usually insist on *regular* parametrizations for which the derivate (tangent velocity vector!) at every time t is *non-zero*.

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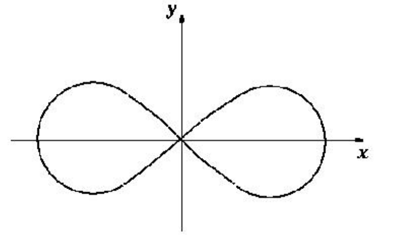
Example: A circle of radius 1 around the origin in the plane can be parametrized

regularly $\gamma : (0,T) \to \mathbf{R}^2$,
$\gamma(t) = (\cos t, \sin t),\ T > 2\pi$;

non-regularly $\bar{\gamma} : (0,T') \to \mathbf{R}^2$,
$\bar{\gamma}(t) = (\cos t^3, \sin t^3),\ T' > \sqrt[3]{2\pi}$.



Lemniscate: $(x^2+y^2)^2 - (x^2-y^2) = 0$

Two parameterizations can describe the *same* geometric curve (that is traversed at different – and varying – speeds). This is the case when one parametrization is a *reparametrization* of the other – you change the "run of time" in a controlled way. To get a unique parametrization, one insists very often on using *unit-speed* parametrizations (or *parametrization by arc length*, cf below).

The *length* of a curve can be calculated by integrating the speed (norm of the velocity vector) along the time interval used for the parametrization. A unit-speed parametrization has the property that one has moved t units of length during t units of time.

Plane curves as level sets

Plane curves have an alternative description: as *level set* of a *smooth* real function on an open set in the plane.

Example: A circle around the origin is the level set of the function $F : \mathbf{R}^2 \to \mathbf{R}, F(x,y) = x^2 + y^2$ (at a positive level).

The inverse (or implicit) function theorem from the analysis course can be applied to show that the two methods for representing curves are equivalent, at least locally: Level sets of smooth functions with *non-vanishing gradient* have a regular parametrization (at least, small pieces of such a level set). On the other hand, a curve with a regular parametrization is the level set of some smooth function (again, at least locally). The proof of the equivalence of the two points of view relies on the two above-mentioned theorems.

Since we will apply these theorems (first of all for the purpose of establishing equivalence of two ways of describing *surfaces*) later on, you are invited to spend some time to make sure you understand what these two theorems tell you.

References

We refer to the literature list:

AP Chapter 1 (with omission of ch. 1.4)

FR Chapter 1.1, 1.2

Wikipedia Curves in differential geometry

Wikipedia Carl Friedrich Gauss

Mac Tutor Gauss biography

Applets

Please experiment with the applets you find in

VIDIGEO A geometric laboratory, in particular applets 3 – 8.

Banchoff Differential Geometry, in particular *Parametrized Space Curve* and *Arclength and arclength reparametrization*

Implicit Plotter Jeb Implicit Plotter

Exercises

As you all know, to learn a new subject requires to look at samples of good examples and on training of essential techniques. This is why you are invited to invest time and energy in the solution of exercises – every single time! You may of course choose exercises from the text book (or also other sources) at your own taste. But I will make suggestions for every single activity, most often from the text book. Please make sure that you train exercises from *different* sections.

You may have a look at hints and, if necessary, solutions in the back part of the book, but only after having tried to find your own way first!

Suggestions

Ch. 1.1 2[1], 3, 5[2], 9[3]

Ch. 1.2 3[4]

Ch. 1.3 1, 4[5].

Ch. 1.5 1[6]

Next activity

Thursday, September 8, 8:15 – 12:00.
Type 1.

Curvature and torsion. Frenet frames. Frenet's equations. Towards the fundamental theorem of curve theory.

[1] (i): Use sinh, cosh
[2] Read Example 1.1.4. Try VIDIGEO
[3] VIDIGEO again. It is tacitly assumed that the curve is plane.
[4] Differentiate $\gamma(\theta) = (r(\theta)\cos\theta, r(\theta)\sin\theta)$.
Unit speed is challenging: Show that the curve is a circle with radius $\frac{1}{2}$ and center $\frac{1}{2}(-\sin\alpha, \cos\alpha)$.
[5] See the proof of Prop. 1.3.4
[6] Jeb Implicit Plotter

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