Schedule

Type 1

8:15 – 10:00, G5-109: Short recapitulation. Two lecture slots.

10:00 – 12:00: Exercise session in group rooms. Lecturer circulates and offers assistance.

Recap. Perspectives

Regular parametrizations, arc length, unit-speed parametrization.
Plane curves as level sets (implicitly given, as solution of an equation).

Lectures

Aims. Content

Unit-speed parametrization: When we work with curves in theory, it is useful to think of them as given by a *unit-speed* (=arc length) parametrization. Only in few examples, does one have closed formulas for such a parametrization, but that does not hamper us theoretically.

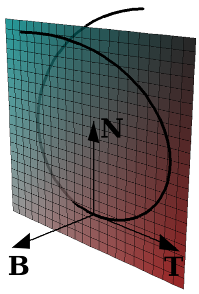
Given a *unit-speed* parametrization, its derivative defines a *vector field* along the curve consisting of *unit* vectors at every point. In other words, the vector field of *unit tangent vectors* $\mathbf{t}(s)$ is not mixed with information on varying speed and contains only geometric information.

Principal normal vectors: Now, analysis enters the scene: Differentiating the equation $1 = \mathbf{t}(s) \cdot \mathbf{t}(s)$ yields: $0 = \mathbf{t}(s) \cdot \dot{\mathbf{t}}(s)$.
Given a unit speed parametrization $\gamma(s)$, this equation shows, that the vector $\dot{\mathbf{t}}(s) = \ddot{\gamma}(s)$ is perpendicular to the tangent vector
$\mathbf{t}(s) = \dot{\gamma}(s)$. When $\dot{\mathbf{t}}(s) \neq 0$, the corresponding unit vector $\mathbf{n}(s) = \frac{\dot{\mathbf{t}}(s)}{\|\dot{\mathbf{t}}(s)\|}$ is the so-called *principal normal vector* at $\gamma(s)$.

Curvature: The numerical rate of change $\| \dot{t}(s) \|$ of $\mathbf{t}(s)$ (with respect to arc length) tells us how quickly the tangent changes direction. This piece of information is called the *curvature* κ of the curve at a given point. The curvature is in fact related to the *radius of curvature* ρ (radius of the *osculating = best approximating* circle) at the same point by the equation $\kappa\rho = 1$. At least when $\kappa \neq 0$.

For a *plane* curve, this concept can be visualized by the *turning angle* φ between the tangent vector $\mathbf{t}$ and the constant vector $\mathbf{e}_1$. Curvature can then be given a (positive or negative) sign; the sign tells us whether the curve turns clockwise (negative curvature) or counter-clockwise (positive curvature) at any given point. The integral over signed curvature yields the progress of the turning angle - it must be an *integer* multiple of 2π for a *closed* curve.

Moving frame and torsion: The next concept is that of a *moving frame*, a coordinate system following the curve. For a plane curve, it consists of the unit tangent vector $\mathbf{t}$ and its "hat" vector $\mathbf{n} = \hat{\mathbf{t}}$.



For a space curve with non-vanishing curvature it consists of three vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$ with $\mathbf{n}$ the principal normal vector and $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ the *binormal vector*. The rate of change (with respect to arc length) of all two, resp. three vectors is interesting; moreover, for a space curve it gives rise to the notion of *torsion* that measures how quickly the curve deviates from the *osculating* (= best approximating) *plane* spanned by $\mathbf{t}$ and $\mathbf{n}$.

Frenet-Serret equations: Expressing the derivatives $\dot{\mathbf{t}}, \dot{\mathbf{n}}, \dot{\mathbf{b}}$ in terms of the moving frame $\mathbf{t}, \mathbf{n}, \mathbf{b}$ leads to the *Frenet-Serret equations*: Let $F(s)$ denote the matrix

$[\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)]$ with the variable Frenet frame as column vectors – an orthogonal (hence invertible) matrix. Its differential $\dot{F}(s)$ can thus be expressed in the form $\dot{F}(s) = F(s)A(s)$ for some matrix $A(s)$. It turns out that this matrix is necessarily *skew-symmetric* with curvature $\kappa(s)$ and (in 3D) *torsion* $\tau(s)$ as the essential entries. Note that this matrix equation can be perceived as a system of four, resp. nine ordinary differential equations; important for the sequel!

Formulas for curvature and torsion: We shall also derive formulas for the curvature and the torsion of a curve given by an *arbitrary regular* parametrized curve – using the usual formulas for derivatives of dot and wedge products in a clever way. Likewise formulas determining the principal normal and binormal vectors. This is achieved without resorting to an explicit reparametrization by arc length.

Review on ODE results

If time permits, we conclude with a review of results about existence and uniqueness for systems of ordinary differential equations; in particular the linear case. You should have a look at them before we embark on the next lecture.

References

AP Ch. 2.1, 2.2, 2.3 (apart from Thm. 2.2.6, 2.3.6)

FR Ch. 1.3.1 – 4, 1.4.1 – 2.

HC Horia Cornean, Notes for Analyse 1 and Analyse 2, ch. 6 (Om eksistens og entydighed af lsninger til sdvanlige diff.ligninger.)

Wikipedia Differential geometry of curves
Frenet-Serret formulas
Initial value problem

Applets

See the course home page.

Exercises

VIDIGEO Experiment with the geometric laboratory, in particular the moving frames and the curvature applets. Use them to elucidate exercises involving concrete parametrizations.

Implicit functions: Investigate the maps

1. $h : \mathbf{R}^2 \to \mathbf{R}, h(x, y) = xy$
2. $k : \mathbf{R}^3 \to \mathbf{R}, k(x, y, z) = xyz$ in light of the *implicit function theorem*. Where are the conditions satisfied, where not? What can you conclude about solutions of equations $xy = c,\ c \in \mathbf{R}$, resp. $xyz = c,\ c \in \mathbf{R}$? How does $c = 0$ differ from the other cases?

ch. 2.1 1(ii), 1(iv), 2[1]

ch. 2.2 1[2], 2, 6[3]

[1] Try (t, t^3, t^4).
[2] Differentiate the equations: $\mathbf{t} \cdot \mathbf{n} = 0$ and $\mathbf{n} \cdot \mathbf{n} = 1$.
[3] Understand the hints on p. 412.

Next activity

September 13, 8:15 – 12:00.
Type 3.
Lecture on the fundamental theorem of curve theory (The curvature function together with the torsion function determine a space curve up to a rigid motion; for plane curves, this is the case for the curvature function alone).
AP, Thm. 2.2.6 and 2.3.6.