Schedule

Type: 3

8:15 – 9:25, G5-109: Short recapitulation and lecture.

9:25 – 12:00 Exercise session in group rooms; some help from the lecturer available.

Recap. Perspectives

Curvature and torsion: Definitions and formulas. Frenet-Serret equations.

Lecture

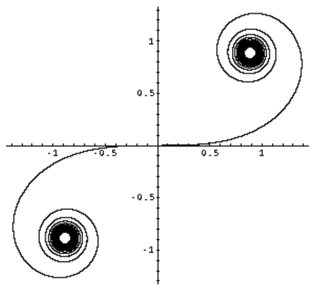
Aims. Content

We have seen, that a curve (independently of the parametrization!) defines a curvature value at each point, and for a space curve also a torsion value whenever the curvature is non-zero. As a consequence, we may associate to a curve a curvature *function* $\kappa(s)$ and, in space, a torsion *function* $\tau(s)$; both are functions of the arc length s.

Fundamental Theorem of Curves (FTC): Reverse question: Is a curve entirely characterized by these two functions? Well, not entirely. It is quite clear that a *rigid motion* – or congruence – in $Iso_+(\mathbf{R}^3)$ – a composition of an orientation preserving orthogonal linear map and of a translation – does not alter either of the two functions. But apart from that, this is what the Fundamental Theorem of Curves (FTC) tells us:

Start with smooth functions $k(s)$ and $t(s)$ defined on an open interval I.[1] Then there is a smooth curve γ which – in unit speed parametrization – has curvature function $\kappa(s) = k(s)$ and torsion function $\tau(s) = t(s)$. Moreover, this curve is well-defined up to a rigid motion.
How can one prove that result?

Plane curve: One integrates the function $k(s)$ and obtains a "turning angle" function $\varphi(s)$ – up to a constant of integration determining a start direction. The function $\varphi(s)$ determines the "moving tangent" function $\mathbf{t}(s)$. Integrating this function yields a unit speed parametrization $\gamma(s)$ – up to a constant of integration that determines the start point.



A *clothoid*: a plane curve with $\kappa(s) = s$.

Space curve: The situation is slightly more complicated: You need to consider the (matrix differential) equation

$$\dot{F}(s) = F(s)A(s)$$

– with $F(s)$ the (unknown!) moving frame matrix, and $A(s)$ the (known!) skew-symmetric matrix with curvature and torsion coefficients $k(s)$ and $t(s)$ – as a *system of ordinary differential equations*. This system consists of nine equations in nine variables, three for each of the vectors $\mathbf{t}, \mathbf{n}, \mathbf{b}$. That system of equations is *linear*. Using existence and uniqueness of solutions of systems of linear ODEs, given (special orthogonal) initial conditions $\mathbf{t}_0, \mathbf{n}_0, \mathbf{b}_0$, there is a

[1] In the plane case, forget about $t(s)$. In the space case, demand that $k(s) > 0$ for all s.

unique solution to these equations, and that solution is defined on the entire domain interval I of the functions k and t.

Next, we check that the solutions $\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s)$ *remain special orthogonal for all* s. For this to be true, it is essential that the matrix $A(s)$ is *skew-symmetric* for all s. Finally, we integrate $\mathbf{t}(s)$ from a given start point (fixing the constant of integration) to obtain a unit speed parametrized curve $\gamma(s)$. Using the Frenet-Serret equations for γ, we check that $\kappa(s) = k(s)$ and that $\tau(s) = t(s)$.

References

FR Ch. 1.3.5 and 1.4.3

AP Thm. 2.2.6 and 2.3.6

Wikipedia Fundamental theorem of curves

Swann A. Swann, On the Fundamental Theorem for Curves in Space

Applets

Curves in Space

Exercises

AP, p. 51 Read Prop. 2.3.5 and its proof. Relate the result to the Fundamental Theorem for Curves(FTC).

AP, p. 49 Check the calculations in Example 2.3.2, p. 49 and continue with Exercise 2.3.2, p. 53.[2]

AP, p. 54 Ex. 2.3.5, 2.3.4[3]

– How do curvature and torsion functions change if you reverse the orientation of the curve? If you reflect a curve in upper half space in the XY coordinate plane?

AP, p. 44 Ex. 2.2.6 (from the previous sheet)

Next Activity

September 15, 8:15 – 12:00.
Type 4.

Main duty: Work out and give a short lecture on the FTC.

[2] Apply FTC!
[3] Hint on p. 402. Another option: Generalise the method described in the proof of the FTC during the lecture and in **FR**.