Schedule

Type 1

8:15 – 10:00 Short repetition and lectures in G5-109.

10:00 – 12:00 Exercise session in group rooms.

Repetition. Perspectives

Definition of a topological and of a smooth surface via charts (surface patches). Level surfaces.

Lectures

The lectures will be dominated by a bunch of applications of the inverse function theorem from mathematical analysis with the aim to elucidate the properties of surfaces and their parametrizations.

Fiddling with parametrizations

Composing a regular chart with a local plane diffeomorphism provides us with an alternative regular chart – a *reparametrization*, comparable with a change of coordinates in Linear Algebra.

The main subtlety in the definition of a surface patch σ is that it *does not make sense* to require that the inverse map σ^{-1} is *smooth*[1] . This difficulty can be overcome using a sequence of results all relying on the inverse function theorem from analysis.

For the first result, we will make use of the three orthogonal projections (preserving two of the coordinates) $\pi_1, \pi_2, \pi_3 : \mathbb{R}^3 \to \mathbb{R}^2$.

Lemma. Given a surface patch $\sigma : U \to S \subset \mathbb{R}^3$.

1. After restriction (to a suitable open subset $\tilde{U} \subseteq U$), at least one of the maps $\pi_i \circ \sigma,\ 1 \leq i \leq 3$, is a (plane) diffeomorphism onto its image.

2. After restriction (to a suitable open subset $\tilde{U} \subseteq U$), the inverse σ^{-1} is the restriction of a smooth map $F : V \to \tilde{U}$ with $V \subset \mathbb{R}^3$ open.

The lemma does not say that σ^{-1} is smooth - that still does not make sense. But σ^{-1} is *the restriction of* a genuine smooth map $F : V \to \mathbb{R}^2$, with $V \subset \mathbb{R}^3$ is open and smooth is defined as it was in analysis 1.
The lemma is proven as an easy consequence of the *inverse function theorem*. It has several important

Consequences The Lemma has more important consequences:

Transition functions As far as possible we are going to replace calculations and definitions on the surface by calculations and definitions in the charts, ie, by *2D*-calculations. As in a physical atlas, one spot may be represented in more than one chart. Therefore, it is important to investigate the transition functions between charts – in order to transfer and extend the results of calculations. It is absolutely essential to check that these transition functions (between two 2D open sets) are *smooth diffeomorphisms*.

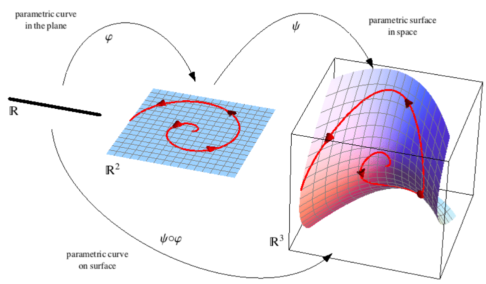
Graph coordinate systems Given a surface parametrization $\sigma : U \to S \subset$

[1] Since σ^{-1} is defined on an open subset of S, not of $\mathbb{R}^3$

$\mathbb{R}^3$. After restriction (to a suitable open subset $\tilde{U} \subseteq U$), there is a *reparametrization diffeomorphism* $\Phi : \bar{U} \to \tilde{U}$ such that $(\sigma \circ \Phi)(x,y) = (x,y,f(x,y))$ and $f : \bar{U} \to \mathbb{R}^3$ is smooth.

Loosely speaking, this property tells you that, locally, every surface can be viewed as the graph of a smooth function defined on an open subset of the plane (the xy, xz or yz plane).

Curves Any smooth curve $c : I \to S \subset \mathbb{R}^3$ can be written (locally) as the composition of a surface patch $\sigma : U \to S$ with a smooth curve $\tilde{c} : I \to U \subset \mathbb{R}^2$.



This property will allow us later to identify the tangent space to a point on a surface in terms of (the differential of) a surface patch.

Smooth maps

A considerable part of the exploration of smooth surfaces proceeds via maps *into* or defined *on* the surface; usually, these maps enjoy smoothness properties that we will have to explain. There are (at least) three instances to discuss:

1. (Smooth) curves on a surface as (smooth) maps from an interval into the surface; cf above.
2. (Smooth) maps from a surface to $\mathbb{R}^m$ and in particular to the real line.
3. (Smooth) maps between two surfaces.

In case 1, the definition is easy: $\gamma : I \to S$ is also a curve in $\mathbb{R}^3$ and smooth means the same as in analysis.

In cases 2 and 3, one combines the maps with parametrizations to define what smoothness means – and then of course one has to check that the definition does *not* depend on the choice of parametrizations.[2]

References

AP Ch. 4.2 – 4.3

FR Ch. 2.4 – 2.5

Wikipedia Differential geometry of surfaces

Applets

Many interesting interactive online illustrations concerning curves and surfaces can be found here.

[2]It turns out that the inverse of a chart, σ^{-1} is a smooth map from (a subset of) the surface to $\mathbb{R}^2$ in the sense of case 2. But you are not allowed to say that before the definition has been established and seen to make sense...

Exercises

Demo Software Get acquainted with some of the applets that were designed together with a text book by Banchoff and Lovett here; in partiuclar those for Chapter 5. I recommend in particular Example 5.1.3 and 5.1.7.

Regularity Let $\sigma = (\sigma_1, \sigma_2, \sigma_3) : U \to \mathbb{R}^3$, $U \subset \mathbb{R}^2$ open, denote a smooth map. Show that the following are equivalent at a point $(u_0, v_0) \in U$:

1. The vectors $\sigma_u(u_0, v_0), \sigma_v(u_0, v_0)$ are linearly *independent*.
2. The Jacobi matrix (the differential) $D\sigma_{(u_0,v_0)}$ has rank 2.
3. The linear map $D\sigma_{(u_0,v_0)} : \mathbb{R}^2 \to \mathbb{R}^3$ is *injective*.
4. $\sigma_u(u_0, v_0) \times \sigma_v(u_0, v_0) \neq \mathbf{0}$.
5. At least one of the three 2×2 subdeterminants of the matrix $D\sigma_{(u_0,v_0)}$ is non-zero.

Level surfaces [AP] 5.1.1 (p. 96).

Ruled surfaces. Surfaces of revolution Work yourself through [AP], ch. 5.3.

Tube surfaces This is [AP], exc. 4.2.7 (p. 81).

Next activity

Time Tuesday, September 27, 8:15 – 12:00 At NJV14 4-117.

Type 3

Content Smooth maps and (local) diffeomorphisms. Tangent planes and normal vectors to a surface. Orientability.

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