Schedule

Type 3

8:15 – 9:30 Short repetition and lecture in NJV 14, 4-117

9:35 – 12:00 Exercise session in group rooms; some help from the lecturer available.

Repetition. Perspectives

Parametrizations of surfaces and their inverses. Transition functions.
Curves on surfaces.

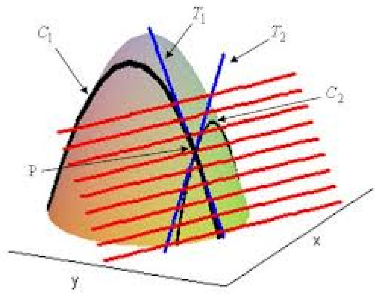
Lecture

Smooth maps between surfaces

How can one check/define whether a map f from a surface S_1 to a surface S_2 is *smooth* at $p \in S_1$? Compose it with a chart for S_1 around p and the inverse of a chart for S_2 around $f(p)$ and check that the composition is smooth.

Is this definition *consistent*? You need to find out that a different choice of charts produces the same answer. This can be checked using transition functions.

A *diffeomorphism* between two surfaces is a smooth map with a smooth inverse. A *local* diffeomorphism needs not have a global inverse, it is enough to ensure that the map has restrictions to open sets (around every point) on which it is has a smooth inverse.



Tangent plane

The *tangent plane* T_pS to a surface S at a point $p \in S$ consists of all tangents $\dot{\gamma}(0) \in \mathbb{R}^3$ to curves γ on S with $\gamma(0) = p$. We have to show that $T_p(S)$ is a 2-dimensional subspace of $\mathbb{R}^3$. Given a surface parametrization $\sigma : U \to \mathbb{R}^3$ with $\sigma(q) = p$ it agrees actually with the subspace $D\sigma_q(\mathbb{R}^2) < \mathbb{R}^3$. And this is 2-dimensional, since $D\sigma_q$ is injective.[1]

The differential of a smooth map

A smooth map $f : S_1 \to S_2$ induces linear derivatives – or differentials
$D_pf : T_pS_1 \to T_{f(p)}S_2$
between tangent spaces at corresponding points. The recipe for the calculation of $D_pf()$ of a tangent vector $\in T_pS_1$ is as follows:
Represent by a curve c_1 on S_1 (with $c_1(0) = p$ and $\dot{c}_1(0) = $). Determine the composite curve $c_2 = f \circ c_1$ on S_2 (with $c_2(0) = f(p)$) and define $D_pf() := \dot{c}_2(0)$.

You have to check that

- the definition is *consistent*, ie, independent of the choice of the particular curve c_1 representing
- the definition leads to a *linear* map D_pf betweeen the tangent spaces.

[1] This is required in the definition of a chart

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Diffeomorphisms and even local diffeomorphisms have the property that the derivative is *invertible* – as a linear map – at every point $p \in S_1$.

Normals and orientability

A (tangent) plane in 3D-space comes with a normal line. Such a line is *not* directed; there are two choices of *unit normal* vectors. Using a surface patch σ for the surface in consideration, these unit normal vectors can easily be calculated as $\mathbb{N}_\sigma = \pm \frac{\sigma_u \times \sigma_v}{\|\sigma_u \times \sigma_v\|}$. The choice of one of the two normal directions induces an *orientation* on the tangent space (perpendicular to the normal line).

Is it always possible to find a *global* orientation on the entire surface S, ie, to find a *continuous* map associating to every point on S a particular unit normal vector? (BTW: If possible, this defines the Gauss map from S to the two-dimensional sphere S^2). The answer is no, in general, and a prominent counter-example is the Möbius band.

A surface with a continuous choice of a unit normal vector is called *orientable*. A surface is orientable if and only if one can find an atlas consisting of surface patches with the property that the determinants of the Jacobians of *all* transition functions are *positive* at every point.

References

AP Ch. 4.3 – 4.5

Wikipedia Tangent space

Wikipedia Orientability

Exercises

AP, p. 85 4.3.1, 4.3.2[2]

Smooth maps as restrictions Given two surfaces S_1, S_2 and a smooth map $F : V \to \mathbb{R}^3$ with $V \subseteq \mathbb{R}^3$ open, $S_1 \subset V$ and $F(S_1) \subseteq S_2$. Show that the restricted map
$G : S_1 \to S_2,\ G(p) = F(p), p \in S_1$
is a smooth map (between the two surfaces).
Use this fact to construct a diffeomorphism F between a 2-dimensional unit sphere S^2 and an ellipsoid (given by the equation in 5.2.2(i) on p. 98).

AP, p. 89 4.4.1(i), 4.4.4.

Tangent spaces to level sets Let $f : \mathbb{R}^3 \to \mathbb{R}$ denote a smooth map, let $S := \{(x,y,z) \in \mathbb{R}^3 | \ f(x,y,z) = c\}$ for some real number c. Assume that $\nabla f(x,y,z) \neq \mathbf{0}$ for each $(x,y,z) \in S$ so that S is a smooth surface.

1. Given $p \in S$. Show that the tangent space T_pS coincides with the kernel (the null space) of the derivative
$D_pf : \mathbb{R}^3 \to \mathbb{R}$, ie, with the orthogonal complement of the gradient vector ∇f at p.
2. Calculate the tangent spaces to the unit sphere S^2 at the North Pole $N : (0,0,1)$ and to the ellipsoid E (5.2.2(i), p. 98) at $Q : (0,0,r)$.
3. Determine the differential $D_NF : T_nS^2 \to T_QE$ at the North Pole of the map F for the diffeomorphim F from the unit sphere to the ellipsoid constructed previously.

Möbius band Draw a Möbius band (Example 4.5.3, pp. 90 – 92) in Banchoff applets using the parametriztion σ on p. 91 and rotate it in 3D-space. Or glue a strip of paper to produce a Möbius band. Then try Exercise 4.5.1 on p. 93 in [AP].

Next Activity

Date Thursday, September 29, 8:15 – 12:00.

Type 4

Content Primarily: Work out and try out a short lecture on the topic:
Smooth maps between surfaces and their derivatives.

[2]You might wish to check out Example 4.3.2, p. 84 to start with.

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