Schedule – Type 2

8:15 – 10:00 Lectures in lecture room G5-109

10:00 – 11:30 Exercises in the group rooms

11:30 – 12:00 Questions and (hopefully) answers from the lecturer

Lecture

Repetition and Perspectives

Differentiable maps and their derivatives. Normal vectors and orientations.

Measurements on surfaces: Arc length and first fundamental form

The length of a curve on a surface can of course be determined without reference to the surface: as the integral over the length of the velocity vectors seen as 3D-vectors. But calculations on a surface should be done via parametrizations. In fact, this will be a point later. How much information is revealed by just measuring *intrinsically*, i.e., without sticking your head up to look around.

Now, all velocity vectors are contained in tangent spaces to the surface, and therefore, they are linear combinations of the basis vectors provided by a parametrization. If $\gamma : I \to S$ and $\gamma(t) = p$, then $\gamma'(t)$ is a linear combination $a\sigma_u(\mathbf{q}) + b\sigma_v(\mathbf{q})$ where $\{\sigma_u(\mathbf{q}), \sigma_v(\mathbf{q})\}$ is the basis of the tangent space $T_\mathbf{p}S$, $\mathbf{p} = \sigma(\mathbf{q})$, corresponding to σ a chart. And clearly $\gamma'(t) \cdot \gamma'(t)$ may be calculated from the coefficients a, b, i.e. the vector $[a, b]$, and information from σ. But how? It is not just the usual inner product of $[a, b]$ with itself.

To this end, one restricts the usual inner (scalar) product on $\mathbb{R}^3$ to a tangent space $T_\mathbf{p}S$. This produces the *first fundamental form* on that tangent space:

$$< \mathbf{v}, \mathbf{w} >= \mathbf{v} \cdot \mathbf{w}, \ \mathbf{v}, \mathbf{w} \in T_\mathbf{p}S.$$

With respect to the basis from above, the first fundamental form can be expressed via its *Gram matrix* $\mathbf{B} = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$ with smooth functions E, F, G of two variables as coefficients, i.e.:

$$E = \sigma_u \cdot \sigma_u, \ F = \sigma_u \cdot \sigma_v, \ G = \sigma_v \cdot \sigma_v.$$

For general vectors $\mathbf{v}, \mathbf{w} \in T_\mathbf{p}S$ expressed in the basis for $T_\mathbf{p}S$ as $\mathbf{v} = v_1\sigma_u + v_2\sigma_v$ and $\mathbf{w} = w_1\sigma_u + w_2\sigma_v$ the first fundamental form can then be calculated as

$$< \mathbf{v}, \mathbf{w} >= [v_1 v_2]\mathbf{B} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$

This allows to express several geometric data in terms of the first fundamental form:

- The *length* of a curve $\gamma(t) = \sigma(u(t), v(t))$ (within a surface patch) is equal to

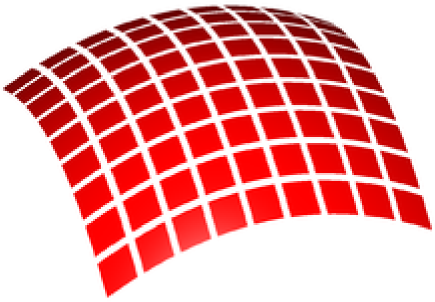
  $$\int_a^b \sqrt{E\, u'(t)^2 + 2F\, u'(t)v'(t) + G\, v'(t)^2}\, dt.$$

  Plug in suitable start and end values a and b for t.

- The *angle* α between two intersecting curves with tangent vectors $\mathbf{v}$, resp. $\mathbf{w}$ is given by the formula $\cos\alpha = \frac{<\mathbf{v},\mathbf{w}>}{\sqrt{<\mathbf{v},\mathbf{v}><\mathbf{w},\mathbf{w}>}}$; for an expression in local coordinates cf. formula (6.7) on p. 133 in the text book.

- The *area* of $\sigma(D) \subset \sigma(U)$ of the image of an integrable region $D \subset U$ is given by

  $$A(D) = \iint_D \sqrt{EG - F^2}\, du dv.$$



The integrand $\sqrt{EG - F^2}$ should be interpreted as a (variable) magnification factor.

The last entry has to be seen as a (well-motivated) *definition* – that can be seen to be independent of the choice of parametrization σ.

Isometries

How can one characterize that a smooth map $f : S_1 \to S_2$ *preserves arc lengths*, i.e., distance along curves within the surface (and then also angles and areas)?
Definition. Let S_1, S_2 denote surfaces with first fundamental forms $<,>_1$ and $<,>_2$. A smooth map $f : S_1 \to S_2$ is a *local isometry* if $< \mathbf{v}, \mathbf{w} >_1 = < D_\mathbf{p} f(\mathbf{v}), D_\mathbf{p} f(\mathbf{w}) >_2$ for all $\mathbf{v}, \mathbf{w} \in T_\mathbf{p} S_1$ and all $\mathbf{p} \in S_1$.

This is the same as to say that the linear map $D_\mathbf{p} f : T_\mathbf{p} S_1 \to T_{f(\mathbf{p})} S_2$ is an isometry of vector spaces with inner product.

It is easy to check that, for a local isometry f and any curve γ on S, the length of γ coincides with the length of $f \circ \gamma$. Similarly, angles and areas of surface regions are preserved under local isometries. Moreover, a local isometry is in fact also a local diffeomorphism. Therefore, given a parametrization σ for S_1, possibly after restriction, the composite map $\sigma_2 := f \circ \sigma$ is a regular parametrization for S_2. Hence, with respect to the two parametrizations σ_1 for S_1 and σ_2 for S_2, the first fundamental forms for the two surfaces *coincide coordinatwise*.

Prominent examples of surfaces that are locally isometric to the plane are cylinders, cones and tangent developables (union of tangent lines of a space curve); cf. pp. 129 – 131 of the text book.

References

AP Ch. 6.1 – 6.2, 6.3, p. 133, and 6.4, pp. 139 – 142.

FR Ch. 4.

Wikipedia First fundamental form

Wikipedia Gramian matrix

Encyclopedia of Mathematics First fundamental form

Applet

Check out the Banchoff applets in Chapter 6.1.

Exercises

Parabolic cylinder A surface S is given by a single parametrization $\sigma(u, v) = (u, u + v^2, v + 1)$, $(u, v) \in \mathbb{R}^2$.

1. Find a simple polynomial equation in the three variables X, Y, Z that is satisfied by (the coordinates of) all points on S. This equation can be used to ex-

plain why the surface is called a parabolic cylinder. You may also draw S in a Banchoff applet.

2. Determine a normal vector $\mathbf{N_p}$ and an equation describing the tangent space $T_\mathbf{p}S$ at the point $\mathbf{p} : (1, 2, 0) \in S$.
3. Determine the first fundamental form (in terms of functions E, F, G) at an arbitrary point $\sigma(u, v) \in S$.
4. Determine the length of the curve $\gamma(t) = \sigma(t, t)$, $-1 \leq t \leq 1$ on S – with and without using the first fundamental form.[1]
5. Determine a (double) integral expression for the area of the region $\sigma(Q)$ with Q the square with side length two centered at the Origin.[2]

AP pp. 124 – 125: 6.1.1 (ii) and (iv), 6.1.2, 6.1.4(ii)[3]

AP pp. 131: 6.2.1, 6.2.2

Next activity

Date Thursday, October 6, 8:15 – 12:00.

Type 1

Content Normal curvature and second fundamental form.
Gauss and Weingarten maps.

References (AP), ch. 7.1 – 7.2.
(FR), ch. 5.2 – 5.4.

[1] An integral expression is enough; you do not need to calculate the integral. It is roughly equal to 4.
[2] The numerical value is around 7.19
[3] This is change of coordinates for a bilinear form. For the notation cf. p. 122. J is the Jacobian of a transition function.

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