Schedule

Type 1.

8:15 – 10:00 Short recap and lecture in G5-109.

10:00 – 12:00 Exercise session in group rooms with assistance from the lecturer.

Recap. Perspectives

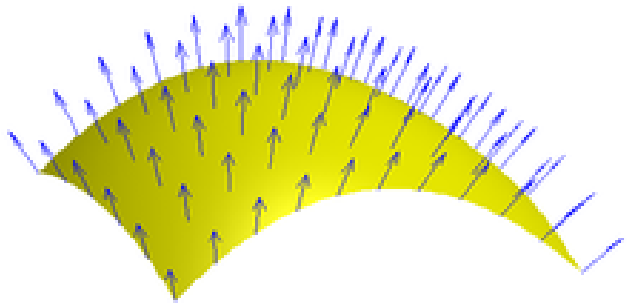
Arc length and first fundamental form. (Local) isometries between surfaces.

Lecture

Gauss- and Weingarten map; the second fundamental form

There are several curvature concepts defined on a surface - curves in the surface and their curvature give an indication, but we have to be careful: A curve in a plane may curve a lot or not at all. However, a curve on a sphere *has* to have non-zero curvature. Moreover, at each point, a surface S may curve differently depending on what direction is chosen, i.e., according to a chosen tangent direction: A tangent vector $\mathbf{t} \in T_{\mathbf{p}}S$ and the normal $\mathbf{N}$ at the point $\mathbf{p}$ span a *normal plane* intersecting the surface in a (space) curve, the so-called *normal section*[1]. The (signed) curvature of that plane curve defines the *normal curvature* in the specified tangent direction $\mathbf{t}$.

As so often, the geometric definition and the intuition behind need to be recast so that they can lead to a systematic understanding and also to calculations. The key to understanding curvature is the *variation* of the unit normal vector $\mathbf{N}$, encrypted in the *Gauss map* $\mathcal{G} : S \to S^2$ – with the unit 2-sphere S^2 as the codomain.[2]



Variation of a map is measured in terms of its *differential*; here: $D_{\mathbf{p}}\mathcal{G} : T_{\mathbf{p}}S \to T_{\mathcal{G}(\mathbf{p})}S^2$. The latter tangent space is perpendicular to $\mathbf{N_p}$ and therefore equal to $T_{\mathbf{p}}S$. This makes the differential above to a linear *self*-map on the (2-dimensional) vector space $T_{\mathbf{p}}S$; moreover, it is *self-adjoint*. We will see later on, that the *eigenvalues* of that map are key to understand normal curvatures of S at $\mathbf{p}$. The spectral theorem from linear algebra makes sure that the tangent space $T_{\mathbf{p}}S$ has an orthonormal eigenvector basis with associated *real* eigenvalues.

A change of signs translates the differential $D_{\mathbf{p}}\mathcal{G}$ into the *Weingarten map* $\mathcal{W}_{\mathbf{p}} = -D_{\mathbf{p}}\mathcal{G}$. The Weingarten map gives rise to the *second fundamental* form on tangent spaces $< \mathbf{v}, \mathbf{w} >_{II} = < \mathcal{W}_{\mathbf{p}}\mathbf{v}, \mathbf{w} >_I$; its Gram matrix (with respect to the basis inherited from a chart σ) is given by $\begin{bmatrix} L & M \\ M & N \end{bmatrix}$ with $L = \sigma_{uu} \cdot \mathbf{N}, M = \sigma_{uv} \cdot \mathbf{N}, N = \sigma_{vv} \cdot \mathbf{N}$.[3]

[1] The definition of a normal section in the textbook on p. 169 is more restrictive than the one we use here.

[2] Remark the analogy to the "turning tangent" map from an interval to the unit circle for curves.

[3] Many references use e, f, g instead of L, M, N.

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Normal and geodesic curvatures

The normal sections are not easy to get a hold of, since they are implicitely defined as an intersection. Hence, it turns out to be a good idea to define the normal curvature not only for normal sections but for general curves on surfaces. At any point, one projects the familiar curvature vector $\ddot{\gamma} = \kappa\mathbf{n}$ (for a unit-speed parametrized curve γ in 3D-space)

- to the normal line yielding the *normal* curvature coefficient $\kappa_n\mathbf{N}$;
- to the tangent plane yielding the *geodesic* curvature coefficient $\kappa_g(\mathbf{N}\times\mathbf{t})$.

It turns out that the geodesic curvature characterizes the particular curve. On the other hand, the normal curvature

- depends only on the tangent direction
- agrees thus with the (signed) curvature of the normal section in that direction (which has geodesic curvature 0 at this point)
- can be calculated as the value of the second fundamental form evaluated (twice) at $\mathbf{t} = \dot{\gamma}$: $\kappa_n(\mathbf{t}) =< \mathbf{t}, \mathbf{t} >_{II}$.

References

AP Ch. 7.1 – 7.3

FR Ch. 5.2 – 5.5

Wikipedia 1 Second fundamental form

Wikipedia 2 Second fundamental form

Wikipedia 3 Gauss map

Wikipedia 4 Curvature of surfaces

Wikipedia 5 Geodesic curvature

Applet

Check out the Banchoff applets in Chapter 6.2 – 4.

Exercises

Plane and sphere For a plane and a sphere of radius R, determine (at every point/direction)

- the normal sections and their curvatures (i.e., the normal curvatures)
- the Gauss- and Weingarten maps

Just think, no calculations are needed in these cases!

AP p. 162, 165: 7.1.1, 7.2.1[4], 7.1.2[5]

First fundamental form This is the same exercise as (AP), 6.1.4(ii). Given two overlapping surface patches σ_1, σ_2 on a surface S and a transition function F with $\sigma_1 = F\circ\sigma_2$.

- Check that the Gram matrix

[4] Compare with applet 6.2.3
[5] Hint: Show that $\mathbf{N}_u = \mathbf{N}_v = 0$ everywhere.

$\begin{bmatrix} E(u,v) & F(u,v) \\ F(u,v) & G(u,v) \end{bmatrix}$ of the first fundamental form for σ_i at $\mathbf{p} \in S$ with $\sigma_i(u,v) = \mathbf{p}$ is given by $(D_{(u,v)}\sigma_i)^T D_{(u,v)}\sigma_i$. Express the Gram matrix of the first fundamental form with respect to σ_1 by a formula using the Gram matrix of the first fundamental form with respect to σ_2 and the Jacobian matrix $D_{(u,v)}F$ of the transition function
$F = \sigma_2 \circ \sigma_1^{-1}$– for (u,v) in the part of the chart corresponding to the overlap[6].

Next activity

Date Tuesday, October 11, 8:15 – 12:00.

Type 3

Content Gaussian and mean curvature; principal curvatures; classification of surface points.
(AP), Ch. 7.3, 8.1 – 8.2.

[6]This is differentiable "change of coordinates" for the first fundamental form