Schedule - Type 3

8:15 – 9:30 Short recap and lecture in G5-109.

9:30 – 12:00 Exercise session in group rooms. Some assistance from the lecturer available.

Midterm evaluation Please remember to fill in the midterm evaluation scheme at Moodle!

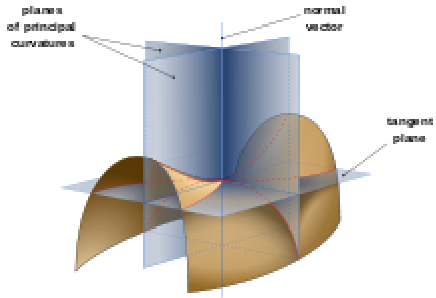
Recap. Perspectives

Normal curvature. Gauss and Weingarten maps. Second fundamental form.

Lecture

Gaussian and mean curvatures

Since information on normal curvature is encoded in the second fundamental form described by the Weingarten map (or matrix $\begin{bmatrix} L & M \\ M & N \end{bmatrix}$ with respect to a parametrization), it is time to find an expression for this matrix in local coordinates; in particular for the determinant and the trace of this matrix. These entities are geometrically significant (and invariant under reparametrization!), the determinant is *Gauss curvature* K and (half of) the trace is the *mean curvature*[1] H; they are easy to calculate from the coefficients of the fundamental forms, cf. Corollary 8.1.3 in the textbook. Up to sign, the Gauss curvature can be interpreted as (differential) area scaling factor for the Gauss map: If the unit normal vector to the surface varies a lot close to a given point, you get a numerically large Gauss curvature; if it is close to constant, the Gauss curvature is numerically small.



Principal curvatures

At every point p on the surface, the Weingarten matrix is symmetric. As a consequence, the tangent plane T_pS admits an *orthogonal* basis of eigenvectors for the Weingarten map, $\mathbf{t}_1, \mathbf{t}_2 \in T_pS$ – the *principal vectors*[2]. The associated eigenvalues are the *principal curvatures*[3] κ_1, κ_2, the *maximal* and the *minimal* normal curvatures at p. As eigenvalues, they are roots of the characteristic polynomial $\kappa^2 - 2H\kappa + K$ and hence:

$$\kappa_{1,2} = H \pm \sqrt{H^2 - K}.$$

From these principal curvatures, one may deduce the normal curvatures in every other tangent direction by *Euler's formula* (Theorem 8.2.4 in the textbook).

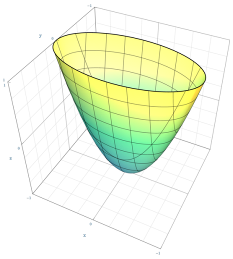
Classification of surface points

Using the principal curvatures, one may classify points on a surface S into four classes. Since normal curvatures are caculated from second order derivatives, they

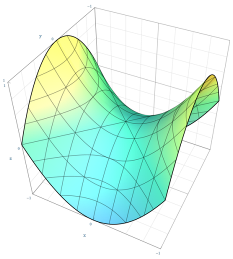
[1] dansk: middelkrumning
[2] dansk: hovedkrumningsvektorer
[3] dansk: hovedkrumninger

indicate which quadratic surface, the surface ”looks like” locally: (not all of them need to occur on a given surface S!) A point p on S is

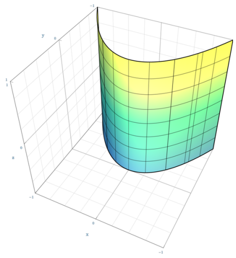
elliptic if $K(p) > 0$, i.e., the principal curvatures at p have the same sign. In this case, the surface curves away from the tangent plane in only one of the two normal directions – locally.



hyperbolic if $K(p) < 0$, i.e., the principal curvatures at p have different signs. In this case, there are directions (given by the principal vectors) in which the surface curves in different normal directions: a saddle point!



parabolic if one of the principal curvatures at p is 0, and the other is not. The surface looks like a parabolic cylinder close to p.



planar if both principal curvatures at p are 0: Second order information cannot tell apart the surface near p from its tangent plane.

References

AP Ch. 7.3, 8.1 – 8.2

FR Ch. 5.4 – 5.5

Wikipedia 1 Geodesic curvature

Wikipedia 2 Gaussian curvature

Wikipedia 3 Mean curvature

Wikipedia 4 Principal curvature

Applets

Check out the Banchoff applets in Chapter 6.4 – 6.5.

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Exercises

AP p. 169: 7.3.3[4].

AP p. 185 – 186: 8.1.1, 8.1.2 (helicoid), 8.1.5[5], 8.1.6[6], 8.1.8, 8.1.9 (nice geometric application of 8.1.8)

Asymptotic vectors A vector $\mathbf{0} \neq \mathbf{t} \in T_pS$ is called *asymptotic* if $\kappa_n(\mathbf{t}) = 0$. Show:

1. T_pS contains an asymptotic vector $\Leftrightarrow K(p) \leq 0$.
2. Let $W(p) = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ denote the Weingarten matrix at p with respect to a coordinate patch and assume $a \neq 0$. The vector $\mathbf{t} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \in T_pS$ is asymptotic if and only if $t_1 = \frac{-b \pm \sqrt{-K(p)}}{a} t_2$. In particular, for $K < 0$, there are two linearly indepent asymptotic vectors.

Next activity

Date Thursday, October 13, 8:15 – 12:00

Type 4

Content First of all: Preparation of a talk on *Curvature notions on a surface*.

[4] Consider a circle of latitude θ = constant, and generalize
[5] dilation = multiplication with factor $c > 0$
[6] Cayley-Hamilton Theorem