Schedule

Type: 4

8:15 – 12:00: *No lecture*. Presentations by group members after preparation. If questions arise, please send an email to the lecturer. They will be addressed at the next session.

Midterm evaluation Please remember to fill in the midterm evaluation scheme on Moodle.

Presentation on curvature notions

You are asked to prepare a short presentation (at most 15 minutes) on *Curvature notions on a surface*. It should include

- Definitions of at least two of the following curvature concepts: normal, principal, Gauss and mean curvature
- An indication of their significance and interconnection
- Recipes for their calculation

Try to make your presentation as self-contained as possible. Illustrative drawings may help to clarify the main ideas.

You are invited to elect one or two group members who are willing to talk in front of the others, and at least one whose task it is to critize – in a constructive way – the presentation with respect to content, (possible) mistakes/misunderstandigs and explanatory "performance". I would be pleased to attend some of the presentations and to give feedback, as well.

As your main sources, you may choose one of the references mentioned on the lecture plan for Activity 10 and 11 or also your lecture notes – or a combination.

Reading instructions

Everything starts with the concept of normal curvature(s) in the tangent directions at a surface point. Geometrically, this can be done via the normal sections of [FR], 5.2. From an algebraic point of view, the definition of normal curvature of a *general* surface curve [AP], 7.3., is preferable, but one needs then to *show* that the normal curvature only depends on the (tangent) direction – certainly not true for curvature as such or for geodesic curvature. This is [AP], Prop. 7.3.3 or [FR], Prop. 5.11.[1]

To understand the variation of the normal curvature in terms of tangent directions, one needs to study the Weingarten map[2] and the second fundamental form; the Gram matrix of that form can easily be calculated using a chart σ. Remark that the coefficients are called L, M, N in [AP] and e, f, g in [FR].

The determinant, resp. $\frac{1}{2}$ the trace of the Weingarten map are called the Gaussian curvature and the mean curvature of the surface at p. They can be calculated using charts as in the proof of Prop. 8.1.2 [AP], or more abstractly and shorter, as in [FR], Section 5.5.

As a consequence of the spectral theorem from Linear Algebra, the tangent plane is spanned by an orthogonal basis

[1] In some references, e.g. Wikipedia, this is called Meusnier's theorem. [AP] reserves this name for a slight generalization, Prop. 7.3.4. For a nice illustration, click here.

[2] Depending on the reference, this is the differential $D_p\mathcal{G}$ of the Gauss map ([FR] and many textbooks) or its negative $-D_p\mathcal{G}$ ([AP]).

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of eigenvectors of the Weingarten map. The eigenvalues are the principal curvatures (which are then shown to be maximal, resp. minimal normal curvatures), and the eigenvectors are the principal directions (well-defined unless one has a repeated eigenvalue). Euler's Theorem ([AP] 8.2.4) shows how to calculate normal curvatures in other directions.

The information given by the *sign* of the Gaussian curvature allows to divide the surface into elliptic, hyperbolic, parabolic and planar points.[3]. This is explained at length in [AP], pp. 192 – 194 applying Taylor's theorem to a (very specific) parametrization. A shorter argument uses just the fact that $K = \kappa_1\kappa_2$ and Euler's theorem.

Supplements and exercises

Leftovers from previous activities

AP, Theorem 8.1.6 An interpretation of (unsigned) Gauss curvature as a differential area scaling factor for the Gauss map. Read and understand the theorem (not necessarily its proof).

AP, Proposition 8.2.9 At an *umbilic* point on a surface, the two principal curvatures agree, and hence so do the normal curvatures in all tangent directions. There are only few surfaces for which *all* points are umbilic: the spheres, including those with radius ∞, ie, planes. Read and uncerstand the proposition (not necessarily its proof).

AP p. 195 8.2.1 (helicoid case), 8.2.2, 8.2.3

Next Activity

Date Tuesday, October 25, 8:15 – 12:00, G5-109.

Type 2

Content Geodesic curves. (AP), ch. 9.1 – 9.2.

[3]For the separation between the two last cases, one needs the eigenvalues explicitly.

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