Schedule

Type 2

8:15 – 10:00 Lectures in lecture room G5-109

11:30 – 12:00 Questions and answers (hopefully) in lecture room G5-109.

Lecture

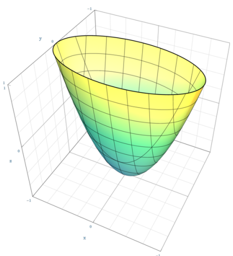
Recap and Perspectives

Curvatures on a surface: normal, principal, Gauss and mean. Definitions, calculations, connections. Normal and geodesic curvature of a curve.

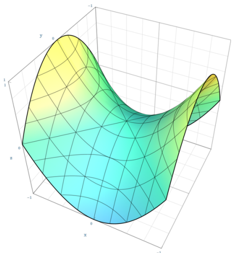
Leftovers concerning curvatures on surfaces.

Classification of surface points Using the principal curvatures, one may classify points on a surface S into four classes. Since normal curvatures are caculated from second order derivatives, they indicate which quadratic surface, the surface "looks like" locally: (not all of them need to occur on a given surface S!) A point p on S is

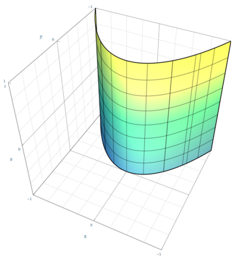
elliptic if $K(p) > 0$, i.e., the principal curvatures at p have the same sign. In this case, the surface curves away from the tangent plane in only one of the two normal directions – locally.



hyperbolic if $K(p) < 0$, i.e., the principal curvatures at p have different signs. In this case, there are directions (given by the principal vectors) in which the surface curves in different normal directions: a saddle point!



parabolic if one of the principal curvatures at p is 0, and the other is not. The surface looks like a parabolic cylinder close to p.



planar if both principal curvatures at p are 0: Second order information cannot tell apart the surface near p from its tangent plane.

Geodesics

A curve on a surface S is called a *geodesic* if it curves *as little as possible*, ie if (equivalently)

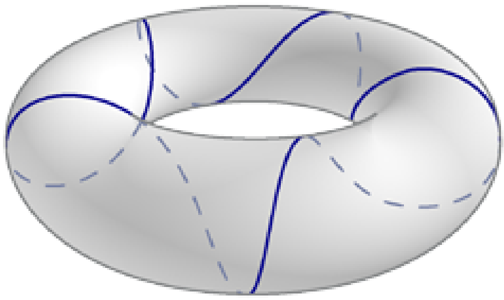
- its curvature at any point p is precisely the minimum given by the normal curvature in the tangent direction
- its geodesic curvature $\kappa_g = 0$ at any point p.

It is easy to see that a geodesic γ always has constant speed $\| \dot{\gamma} \|$.

Easy first examples (in constant speed parametrization):

- Straight lines (if such exist on the given surface)
- Great circles on spheres

How about geodesics on a cylinder (are the lines the only geodesics?) On a torus?



Geodesic differential equations

It turns out that geodesic curves are the solutions of a pair of homogeneous second order differential equations satisfied by the coordinates $u(t), v(t)$ of the curve $\gamma(t) = \sigma(u(t), v(t))$ – given a coordinate patch σ. These *geodesic equations* express that the curvature vector $\ddot{\gamma}$ is perpendicular to the tangent plane everywhere.

To put the geodesic equations into standard form, one applies the so-called *Christoffel symbols* Γ^i_{jk} that arise as coefficients in the expressions of double derivatives of σ in the basis given by σ_u, σ_v, N. It turns out that the Christoffel symbols only depend on the surface's 1st fundamental form (and its partial derivatives).

Theory of differential equations at work

It is in general difficult or impossible to solve the geodesic equations explicitly. Theory shows its value: An initial value for a pair of second order equations consists of a point *and* a velocity vector. We can thus infer from the theorem on existence of solutions to an initial value problem:

There is a *unique* geodesic through *any* given point of a surface with *given velocity vector* at that point.

Since the differential equations are homogeneous (of degree 2), every linear reparametrization $\tilde{u} = ku, \tilde{v} = kv, \ k \neq 0$ of a geodesic yields a geodesic again - with speed multiplied by $|k|$. Hence, we may replace *velocity vector* by *tangent direction* in the above statement.

We conclude that we already know *all* geodesics on the plane and on a sphere.

References

AP Ch. 9.1 – 9.2 (until p. 224), Proposition 7.4.4

FR Ch. 6.3 and 7.1

Wikipedia Geodesic

Applet

Banchoff-Lovett Check out the Banchoff applet in Chapter 8.2.

Visual Geometry Try "geodesic curves" on Curves on Hypersurfaces

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Exercises

Leftovers among AP, 8.2.1 – 8.2.3[1]

AP 8.2.7[2]

AP 9.1.1[3] – 9.1.2

Next activity

Monday, October 31, 8:15 – 12:00.
Type 1.

Isometries and geodesics. Preparations for Theorema egregium: Gauss and Codazzi-Mainardi equations.
Text book: [AP], ch. 9.2.7 – 8, 10.1.

[1] Idea: $W\dot{\gamma}$ and $\dot{\gamma}$ are parallel. Hence the 2×2-matrix with these two column vectors (wrt. $\{\sigma_u, \sigma_v\}$) has determinant 0.

[2] Solve the equation describing the ellipsoid locally with respect to one of the variables. At an umbilic point, the two fundamental forms are proportional.

[3] See exercise 4.1.3; straight lines and appropriate normal sections!