Schedule

Type: 1

8:15 – 10:00: Short repetition and lecture.

10:00 – 12:00: Exercise session in group rooms.

Repetition. Perspectives

Geodesics.
Geodesic differential equations: existence and uniqueness of solutions.

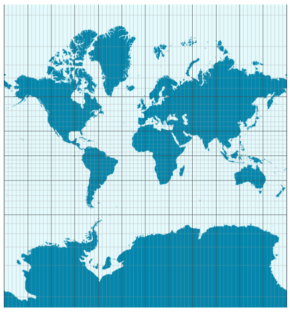
Lecture

Isometries and geodesics

A (local) isometry leaves the first fundamental form invariant. Geodesics are solutions of the geodesic differential equations that can be expressed in terms of the first fundamental form and its derivatives; geodesics are thus *intrinsic*[1] entities.
Consequence: A (local) isometry maps geodesics into geodesics. In particular, the geodescics on a surface that is locally isometric to the plane (a cylinder, a cone...) can be found as images of lines on the plane under the isometry.

The goal: Gauss' Theorema egregium

An isometry will, in general, change normal and also principal curvatures. But the Gaussian curvature is left invariant! In other words, it is an *intrinsic invariant*. That is the content of Gauss' celebrated Theorema egregium[2]. It has the immediate consequence that a spherical surface is not (locally) isometric to the plane: There are no maps (projections) between sphere and plane without distortion![3]



The Mercator projection
preserves angles
not constant scale.

The Gaussian curvature is usually expressed by a formula involving *both* the first *and* the second fundamental form. As in the previous case – geodesics – the Theorema Egregium can be proved if one is able to express the Gaussian curvature by a formula involving the first fundamental form (and its partial derivatives) *only*.

Equations: Codazzi-Mainardi, Gauss

More generally speaking, what relations are there between coefficients of the two fundamental forms? Remember that the Christoffel symbols Γ^i_{jk} can be expressed in terms of the first fundamental form and its derivatives. Hence, if it is possible to express the Gaussian curvature by coefficients of the first fundamental form, the Christoffel symbols and their partial derivatives, the remarkable theorem is proved. It turns out that this can be done (mere magic, it seems!) exploiting the

[1]only dependent on the metric of the surface in terms of the first fundamental form, not on the way the surface is embedded into ambient 3-space
[2]dansk: fremragende, bemærkelsesværdig
[3]Landkort har aldrig konstant målestoksforhold

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fact that one may permute the order of partial derivatives of a local parametrization – in this case of the third(!) order.

Along with the so-called *Gauss equations*, that allow one to express EK, FK and GK in terms of Christoffel symbols and their partial derivatives, there are also relations between the coefficients of the second fundamental form (their partial derivatives) and the Christoffel symbols in terms of the *Codazzi-Mainardi equations*. It turns out, that these two sets of equations together characterize pairs of forms that arise as fundamental forms of surfaces – unique up to direct isometry.

For some of the proofs, you might have to wait until next time.

References

AP Ch. 9.2, pp. 224 – 226; Ch. 10.1 – 10.2 (only Theorem 10.2.1)

FR Ch. 6.4.

Wikipedia 1 Theorema Egregium

Wikipedia 2 Gauss-Codazzi equations

Applet

Check out the Banchoff applets in Chapter 7.

Exercises

AP First note that the definition of a normal section γ in the textbook on p. 169 is far more restrictive than the one given in the course. It requires that the intersecting plane Π is perpendicular to the tangent plane at *every* intersection point.

- Check Corollary 7.3.5 and its consequence Proposition 9.1.6 for these very special normal sections.
- Prove that the vector $\mathbf{u} = \mathbf{u}(s) = N(s) \times \dot{\gamma}(s)$ is the *constant* normal vector to the intersecting plane Π.
- Differentiate the equation $N(s) \cdot \mathbf{u} = 0$; what can you infer about $\dot{N}(s)$?
- Prove that the tangent vector $\mathbf{t}(s) = \dot{\gamma}(s)$ is a *principal* vector for every s[4]. Hence γ is a geodesic *line of curvature* (cf definition in Execrcise 8.2.2 on p. 195).
- Which normal sections are there on a cylinder? (Perhaps you have to wait until having solved Exercise 9.2.1 or 9.2.3).

AP p. 221 – 222: Read and understand Example 9.2.2 concerning geodesics on the 2-sphere S^2.

AP p. 226: 9.2.1[5] – 3.

[4] Show that $\dot{\gamma}(s)$ is an eigenvector of the Weingarten map
[5] Apply a local isometry from the plane to the cylinder

Next activity

Friday, October 29(!), 8:15 – 12:00.
Type 3.

Equations: Codazzi-Mainardi, Gauss.
Theorema Egregium and some of its consequences.
Text book: [AP], ch. 10.1-2.