Schedule

Type: 3

8:15 – 9:30: Short repetition and lecture. G5-109.

9:30 – 12:00 Exercise session in group rooms.
Some help from the lecturer available.

Repetition. Perspectives

Local isometries preserve geodesics and Gaussian curvature (Theorema egregium). Gauss equations, Codazzi-Mainardi equations.

Lectures

Equations: Codazzi-Mainardi, Gauss

What relations are there between coefficients of the two fundamental forms? The *Gauss equations* allow to express EK, FK and GK in terms of Christoffel symbols and their partial derivatives. Remember that the Christoffel symbols Γ^i_{jk} can be expressed in terms of the first fundamental form and its derivatives. Hence, if it is possible to express Gauss curvature by coefficients of the first fundamental form, the Christoffel symbols and their partial derivatives, and this proves the remarkable theorem![1] The Gauss equations come up (mere magic, it seems!) exploiting the fact that one may permute the order of partial derivatives of the third(!) order.

Along with these equations, there are also relations between the coefficients of the second fundamental form (and their partial derivatives) and the Christoffel symbols in terms of the *Codazzi-Mainardi equations*. It turns out, that these two sets of equations together characterize pairs of forms that arise as fundamental forms of surfaces – unique up to isometry. [2]

We show how to reconstruct a surface from the two fundamental forms in a simple example.

Theorema Egregium (TE) and some consequences

The formulas derived for the Gaussian curvature in terms of the Christoffel symbols can be rewritten to yield a formula for K in terms of the coefficients E, F, G of the first fundamental form – and their partial derivatives. The case where $F = 0$ (an orthogonal parametrization) is of particular importance and will be used later on.

We note important consequences of TE: A sphere (with $K > 0$ everywhere) is not locally isometric to the plane (with $K = 0$ everywhere) – and thus every plane map of a region on the earth's surface *must distort distances – even at a small scale*. Likewise (non-round) ellipsoids or tori can not be mapped isometrically onto each other, a sphere or a plane. As an example, we will classify the self-isometries of a helicoid.

[1] The Christoffel symbols themselves are NOT intrinsic geometric invariants. They are not even geometric - they clearly depend on the choice of chart

[2] This is the analogue of the Fundamental Theorem for Curves.

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References

AP Ch. 10.1 – 10.2

FR Ch. 6.4.1

Wikipedia 1 Theorema Egregium

Wikipedia 2 Gauss-Codazzi equations

Applet

Visual Geometry Curves on Hypersurfaces

Exercises

Geodesics Try "geodesic curves" on Curves on Hypersurfaces: Read first the instructions below the applet. Try the sphere (not that interesting) and the torus (much more so!). You can animate the direction in which the geodesics are "fired". What about other surfaces?

AP 10.1.1[3], 10.1.2[4].

AP 10.2.1, 10.2.5.

Next activity

Thursday, November 1, 8:15 – 12:00.
Type 4.

Training session: Theorema Egregium.

[3] Calculate the Weingarten map for such a surface and conclude a connection between normal vector N and the parametrization σ

[4] Calculate Christoffel symbols and check the Codazzi-Mainardi equations

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