Schedule

Type 2

8:15 – 10:00 Lectures in lecture room G5-109

10:00 – 11:30 Exercises in the group rooms

11:30 – 12:00 Questions and answers (hopefully) in lecture room G5-109.

Lecture

Recap and Perspectives

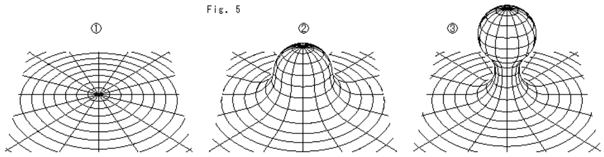
Theorema egregium and consequences.
Expressions for Gaussian curvature.
Isometries between surfaces, extendable to spatial isometries?

The Gauss-Bonnet theorem for simple closed curves

The Gaussian curvature at a point characterizes the local geometry of a surface close to that point. What happens when one integrates (averages) the Gaussian curvature – this amounts to calculating the *total curvature* of the surface? Think a little about the special cases of a sphere with radius R and of a torus before we go on:

To start the investigation of that problem, we first consider integrating the Gaussian curvature over the interior of a *simple closed curve* γ on a surface patch. The result (Theorem 13.1.2 in the text book) relates the total curvature over the interior of γ (a 2D-integral) to the integral of the *geodesic curvature* of the bounding curve γ (a 1D-integral).

Fig. 5



An immediate consequence is the following. Draw a simple closed curve in the plane. The total curvature of the area inside the the curve is zero. Now make a dent (smoothly) in the interior without changing the immediate neighborhood of the curve. The total curvature of the interior is still zero.

The proof relies, apart from some quite lengthy calculations, on two important results that we will use without proving them:

Green's theorem a general analytic theorem relating a 2D-integral of the curl of a 2D vector field over a plane region bounded by a simple (positively oriented) curve to the integral of the vector field along that boundary curve. It generalizes the fundamental theorem of Calculus to 2D and is itself a special case of Stokes' theorem. For a proof, see e.g. the reference [AE] below, or Wikipedia's account.

Hopf's Umlaufsatz of a topological nature calculating the integral of the angular velocity along a closed plane curve.

The lengthy calculations alluded to above includes

- calculating the geodesic curvature in terms of an orthonormal basis of the tangent plane at each point;
- interpreting the integral over the result as a line integral and to use Hopf's Umlaufsatz for one of the terms, and to interpret the other one using Green's Theorem. The latter part is a lengthy calculation in its own right.

References

AE R.A. Adams, Ch. Essex, *Calculus*, Ch. 16.3, pp. 903 – 904

AP A. Pressley, *Elementary Differential Geometry*, ch. 13.1 – 2.

Swann A. Swann, Hopf's Umlaufsatz

Wikipedia Green's theorem

Wikipedia Gauss-Bonnet theorem

Exercises

AP 13.1.1, 13.2.1[1]

Swann Begin to read the account for Hopf's Umlaufsatz

Next activity

December 6, 8:15 – 12:00.
Type 1.

Gauss-Bonnet for compact surfaces.

[1] Use the formula for the Gaussian curvature on a surface of revolution obtained in Example 8.1.4.

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