Schedule

Type 1

8:15 – 10:00 Short recap. Lectures. G5-109.

10:00 – 12:00 Exercise session in group rooms. Lecturer circulates.

Recap. Perspectives

The local Gauss-Bonnet Theorem:

- expressing geodesic curvature in local orthonormal coordinates
- applying Hopf's Umlaufsatz and Green's Theorem
- interpretation of the function to be integrated by way of the Gaussian curvature

Lectures

Gauss-Bonnet for polygons

To piece results together from several coordinate patches without overlap, it is preferable to use curves with edges, so-called curvilinear *polygons*. In this case, the previous Gauss- Bonnet result has to be modified: The angles between adjacent edges meeting in a vertex of the polygon have to be considered, as well. See Theorem 13.2.2 and its Corollary 13.2.3 for *geodesic* polygons. In the case of geodesic polygons, only these angles count! What does that mean for, say, a spherical triangle made of great circle arcs?

Integration on compact surfaces

While integration of a function defined within a region covered by one surface patch can be traced back to integration of a related function on a related region in the plane (2D), a little more thought is necessary to get to a formally pleasing integral over the entire surface.
One way to do that is to extend surface and function to a small normal "collar" around the surface, to apply 3D-integration along the collar and to observe what happens to the integrals when the collar gets smaller and smaller.
Another way is to use so-called partitions of unity functions – that add up to the constant function 1 and with each of them having support (values $\neq 0$) within a coordinate patch.

Gauss-Bonnet for compact surfaces

The global Gauss-Bonnet theorem deals with the *total curvature* along a compact surface – without boundary; ie the integral of Gaussian curvature over the *entire surface*. Somehow surprisingly, it turns out that the result does not depend on *geometric* properties of the surface; only the *topology* counts! In other words, however crazily you might deform a surface continuously (without destroying it), this integral always yields the same result.

The integral may in fact be expressed in terms of a *triangulation* of the surface, a partition of the surface into regions bounded by curvilinear polygons. It is not that easy to show that such a triangulation always exists, but we will take that for granted.
A triangulation comes with a number V of *vertices*, a number E of *edges* and a num-

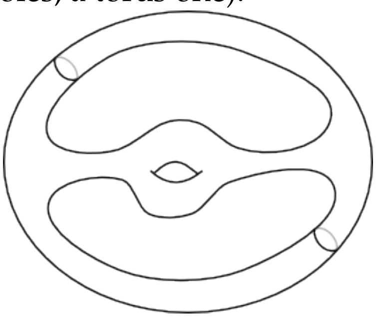
ber F of *faces*. These numbers depend, of course, on the triangulation, but the *Euler characteristic* $\chi(S) = V - E + F$ *does not!*

And the answer is: $\int_S K dA = 2\pi\chi(S)$. The proof proceeds by summing up the contributions from each polygon in a triangulation. The terms arising from (edge) integrals of the geodesic curvature cancel out two by two; the "angle terms" sum up to $2\pi\chi(S)$.

Interpretations and consequences

First of all, the theorem tells you that poking continuous bumps into a surface will change the Gaussian curvature in such a way that the areas in which Gaussian curvature decreases will exactly balance out the areas in which Gaussian curvature increases (in terms of the integral over curvature).

What kind of surfaces are we talking about? It turns out that *every compact orientable surface* is diffeomorphic to a surface T_g "with g holes", that you obtain by glueing g tori (torusses) together (a sphere has 0 holes, a torus one).[1]



The integer g is called the *genus* of that surface. It is not difficult to calculate $\chi(T_g) = 2 - 2g$; in particular, the total curvature is *negative* for $g > 1$. There are models of these surfaces with *hyperbolic* geometry, ie with constant negative Gaussian curvature.

References

AP A. Pressley, *Elementary Differential Geometry*, ch. 13.2 –4.

Cornell Euler characteristic

Wikipedia Gauss-Bonnet theorem

Wikipedia Euler characteristic

Wikipedia Genus

Wikipedia Partition of unity

[1] Everything becomes much more complicated for manifolds of a dimension bigger than 2.

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Exercises

AP, 13.2.1 Try again!

AP, 8.6 Read Chapter 8.6 on compact surfaces. Note in particular, that there are elliptic points on every compact surface.

AP, 13.4 13.4.1, 13.4.2 (Use Theorem 13.4.5 and 13.4.7).

Next activity

December 8, 8:15 – 12:00.
Type 3.

Applications of the Gauss-Bonnet theorem.

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