A Note on Korovkin’s Theorem

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Introduction

This note concerns a general approach to theorems of “Weierstrass type”. Convergence of Bernstein polynomials and many other classical approximation algorithms can be deduced from Korovkin’s theorem below.

We consider the family $C(A)$ of (real- or complex-valued) continuous function on a compact Hausdorff space $A$. A map $U: C(A) \to C(A)$ is called linear if $U(\alpha f + \beta g) = \alpha U(f) + \beta U(g)$ for $f, g \in C(A)$ and arbitrary scalars $\alpha, \beta$. $U: C(A) \to C(A)$ is called positive if $U(f) \geq 0$ whenever $f \geq 0$. Notice that a linear positive map $U: C(A) \to C(A)$ satisfies $U(f) \leq U(g)$ if $f \leq g$ and consequently

$$|U(f)| \leq U(\|f\|) \leq U(\|f\| \cdot 1) = \|f\|U(1) \leq \|f\| \cdot \|U(1)\|,$$

with $\|f\| := \max_{x \in A} |f(x)|$.

Korovkin’s Theorem

Korovkin has shown\(^1\) that for a sequence \(\{U_n\}\) of linear positive maps $U_n: C(A) \to C(A)$, in many cases, uniform convergence $U_n(f) \to f$ follows for all $f \in C(A)$, if it holds for a suitable finite collection of “test functions”. His result is

**Theorem.** Assume that there exist continuous real functions $a_i(y)$ on $A$, $i = 1, 2, \ldots, m$, such that

$$P_y(x) = \sum_{i=1}^{m} a_i(y)g_i(x) \geq 0 \quad (1)$$

for all $x, y \in A$ and that $P_y(x) = 0$ if and only if $x = y$. Then for a sequence \(\{U_n\}\) of linear positive maps on $C(A)$, the (uniform) convergence

$$U_n(g_i) \to g_i, \quad n \to \infty, \quad i = 1, 2, \ldots, m \quad (2)$$

implies that

$$U_n(f) \to f, \quad \text{(uniformly)}, \quad n \to \infty, \quad \forall f \in C(A). \quad (3)$$

**Proof.** First we fix a function $P^*(x) := P_{y_1}(x) + P_{y_2}(x)$, with $y_1 \neq y_2$, and notice that $P^* > 0$

on $A$. We consider some properties of the “polynomials” $P(x) = \sum_{i=1}^{m} a_i g_i(x)$. From (2) we have $U_n(P, x) \to P(x)$ uniformly in $x$ on $A$. We also have

$$U_n(P_y, y) = \sum_{i=1}^{m} a_i(y) U_n(g_i, y) \to \sum_{i=1}^{m} a_i(y) g_i(y) = P_y(y) = 0$$

(4)

uniformly in $y$ since each $a_i(x)$ is bounded on $A$. Finally, notice that

$$0 \leq U_n(1, x) \leq \gamma U_n(P^*, x) \to \gamma P^*(x), \quad \text{(uniformly)},$$

with $\gamma = 1/\min_{x \in A} P^*(x) < \infty$. Therefore there exists $M_0 < \infty$, satisfying $\sup_n \|U_n(1, y)\| \leq M_0$.

We need the following fact. Let $f_y \in C(A)$, $y \in A$, be a family of functions for which $f_y(x)$ is a continuous function of $(x, y) \in A \times A$, and $f_y(y) = 0$ for all $y \in A$. Then

$$U_n(f_y, y) \to 0, \quad \text{uniformly in } y, \quad n \to \infty.$$  

(5)

To prove this, let $B = \{(y, x) : y \in A\}$, and let $\varepsilon > 0$ be given. By continuity of $f_y(x)$ on $A \times A$, each point $p \in B$ has an open neighbourhood $V_p$ in $A \times A$ satisfying $|f_y(x)| < \varepsilon$ if $(x, y) \in V_p$. Then $G = \bigcup_{p \in B} V_p$ is open in $A \times A$, and the complement $F = A \times A \setminus G$ is closed and thus compact. Let

$$a = \min_{(x, y) \in F} P_y(x) > 0, \quad \text{and} \quad b = \max_{(x, y) \in F} f_y(x).$$

Notice that

$$|f_y(x)| \leq \varepsilon + \frac{b}{a} P_y(x), \quad \forall x, y \in A.$$  

(6)

From (6) we deduce that

$$|U_n(f_y, y)| \leq U_n(\varepsilon + \frac{b}{a} P_y(x), y) \leq \varepsilon U_n(1, y) + \frac{b}{a} U_n(P_y, y) \leq \varepsilon M_0 + \frac{b}{a} U_n(P_y, y).$$

However, by (4), there exists $N$ such that $n \geq N$ implies that $|U_n(f_y, y)| \leq \varepsilon (M_0 + 1)$. This proves the claim.

Now, the proof of the theorem follows easily. If $f \in C(A)$ we define

$$f_y(x) = f(x) - \frac{f(y)}{P^*(y)} P^*(x).$$

By (5),

$$U_n(f_y, y) = U_n(f, y) - \frac{f(y)}{P^*(y)} U_n(P^*, y) \to 0,$$  

(7)

uniformly in $y$. We have

$$|U_n(f, y) - f(y)| \leq \left| U_n(f, y) - \frac{f(y)}{P^*(y)} U_n(P^*, y) \right| + \left| \frac{f(y)}{P^*(y)} U_n(P^*, y) - f(y) \right|$$

$$= \left| U_n(f, y) - \frac{f(y)}{P^*(y)} U_n(P^*, y) \right| + \left| \frac{f(y)}{P^*(y)} \right| \left| U_n(P^*, y) - P^*(y) \right|$$

$$\leq \left| U_n(f, y) - \frac{f(y)}{P^*(y)} U_n(P^*, y) \right| + \gamma \|f\| \left| U_n(P^*, y) - P^*(y) \right|$$

Using (7) and the fact that $U_n(P^*, y) \to P^*(y)$, uniformly, the theorem follows. \qed
Applications

Bernstein polynomials

Consider the operator $B_n : C([0, 1]) \to C([0, 1])$ generating the Bernstein polynomial

$$f \mapsto B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f \left( \frac{k}{n} \right), \quad n = 1, 2, \ldots$$

Clearly each $B_n$ is linear and positive. Also notice that $B_n(1) = 1$, $B_n(t)(x) = x$, and $B_n(t^2)(x) = x^2 + \frac{x(1-x)}{n}$. We can therefore use

$$P_y(x) = (x - y)^2 = x^2 - 2yx + y^2 := x^2 g_1(y) - 2xg_2(y) + g_3(y)$$

in Korovkin’s theorem to conclude that $B_n(f) \to f$ uniformly for all $f \in C([0, 1])$.

Trigonometric sums

Consider the Féjer maps $\sigma_n : C(\mathbb{T}) \to C(\mathbb{T})$ given by

$$f \mapsto \sigma_n(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) F_n(x - t) \, dt, \quad F_n(\alpha) = \frac{1}{2(n+1)} \frac{\sin^2 \left( \frac{(n+1)\alpha}{2} \right)}{\sin^2 \left( \alpha/2 \right)}.$$  

Using certain well-known trigonometric identities, one can verify that

$$\sigma_n(f) = \frac{s_0(f) + s_1(f) + \cdots + s_n(f)}{n+1},$$

where $s_k(f)$ is the $k$’th partial sum of the Fourier series of $f$:

$$s_k(f) = \frac{a_0}{2} + \sum_{j=1}^{k} \left( a_j \cos jx + b_j \sin jx \right),$$

with

$$a_j := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad \text{and} \quad b_j := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt.$$  

Notice that $\sigma_n$ is linear and positive. One easily verifies that for $n \geq 1$: $\sigma_n(1) = 1$, $\sigma_n(\cos) = \frac{n}{n+1} \cos x$, and $\sigma_n(\sin) = \frac{n}{n+1} \sin x$. Thus, Korovkin’s Theorem applies with

$$P_y(x) = 1 - \cos(x - y) = 1 - \cos(y) \cos(x) - \sin(y) \sin(x),$$

and $\sigma_n(f) \to f$ uniformly for every $f \in C(\mathbb{T})$.  

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