A Note on Korovkin's Theorem

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Introduction

This note concerns a general approach to theorems of "Weierstrass type". Convergence of Bernstein polynomials and many other classical approximation algorithms can be deduced from Korovkin's theorem below.

We consider the family $C(A)$ of (real- or complex-valued) continuous function on a compact Hausdorff space A. A map $U\colon C(A) \to C(A)$ is called **linear** if $U(\alpha f + \beta g) = \alpha U(f) + \beta U(g)$ for $f, g \in C(A)$ and arbitrary scalars α, β. $U\colon C(A) \to C(A)$ is called **positive** if $U(f) \geq 0$ whenever $f \geq 0$. Notice that a linear positive map $U\colon C(A) \to C(A)$ satisfies $U(f) \leq U(g)$ if $f \leq g$ and consequently

$$|U(f)| \leq U(|f|) \leq U(\|f\| \cdot 1) = \|f\| U(1) \leq \|f\| \cdot \|U(1)\|,$$

with $\|f\| := \max_{x \in A} |f(x)|$.

Korovkin's Theorem

Korovkin has shown[1] that for a sequence $\{U_n\}$ of linear positive maps $U_n\colon C(A) \to C(A)$, in many cases, uniform convergence $U_n(f) \to f$ follows for all $f \in C(A)$, if it holds for a suitable finite collection of "test functions". His result is

Theorem. Assume that there exist continuous real functions $a_i(y)$ on A, $i = 1, 2, \ldots, m$, such that

$$P_y(x) = \sum_{i=1}^{m} a_i(y) g_i(x) \geq 0 \tag{1}$$

for all $x, y \in A$ and that $P_y(x) = 0$ if and only if $x = y$. Then for a sequence $\{U_n\}$ of linear positive maps on $C(A)$, the (uniform) convergence

$$U_n(g_i) \to g_i, \qquad n \to \infty, \quad i = 1, 2, \ldots, m \tag{2}$$

implies that

$$U_n(f) \to f, \qquad \text{(uniformly)}, \quad n \to \infty, \quad \forall f \in C(A). \tag{3}$$

Proof. First we fix a function $P^*(x) := P_{y_1}(x) + P_{y_2}(x)$, with $y_1 \neq y_2$, and notice that $P^* > 0$

[1] Korovkin, P. P. (1957) On convergence of linear positive operators in the space of continuous functions. Dokl. Akad. Nauk SSSR (N.S.) **114**, 961-964; Ch. 1–8.

on A. We consider some properties of the "polynomials" $P(x) = \sum_{i=1}^m a_i g_i(x)$. From (2) we have $U_n(P,x) \to P(x)$ uniformly in x on A. We also have

$$U_n(P_y, y) = \sum_{i=1}^m a_i(y) U_n(g_i, y) \to \sum_{i=1}^m a_i(y) g_i(y) = P_y(y) = 0 \tag{4}$$

uniformly in y since each $a_i(x)$ is bounded on A. Finally, notice that

$$0 \le U_n(1,x) \le \gamma U_n(P^*, x) \to \gamma P^*(x), \qquad \text{(uniformly)},$$

with $\gamma = 1/\min_{x\in A} P^*(x) < \infty$. Therefore there exists $M_0 < \infty$, satisfying $\sup_n \|U_n(1,y)\| \le M_0$.

We need the following fact. Let $f_y \in C(A)$, $y \in A$, be a family of functions for which $f_y(x)$ is a continuous function of $(x,y) \in A \times A$, and $f_y(y) = 0$ for all $y \in A$. Then

$$U_n(f_y, y) \to 0, \qquad \text{uniformly in } y, \quad n \to \infty. \tag{5}$$

To prove this, let $B = \{(y,y) : y \in A\}$, and let $\varepsilon > 0$ be given. By continuity of $f_y(x)$ on $A \times A$, each point $p \in B$ has an open neighbourhood V_p in $A \times A$ satisfying $|f_y(x)| < \varepsilon$ if $(x,y) \in V_p$. Then $G = \cup_{p\in B} V_p$ is open in $A \times A$, and the complement $F = A \times A \backslash G$ is closed and thus compact. Let

$$a = \min_{(x,y)\in F} P_y(x) > 0, \qquad \text{and} \qquad b = \max_{(x,y)\in F} f_y(x).$$

Notice that

$$|f_y(x)| \le \varepsilon + \frac{b}{a} P_y(x), \qquad \forall x, y \in A. \tag{6}$$

From (6) we deduce that

$$|U_n(f_y, y)| \le U_n\big(\varepsilon \cdot 1 + \frac{b}{a} P_y(x), y\big) \le \varepsilon\, U_n(1,y) + \frac{b}{a}\, U_n(P_y, y) \le \varepsilon\, M_0 + \frac{b}{a}\, U_n(P_y, y).$$

However, by (4), there exists N such that $n \ge N$ implies that $|U_n(f_y, y)| \le \varepsilon(M_0 + 1)$. This proves the claim.

Now, the proof of the theorem follows easily. If $f \in C(A)$ we define

$$f_y(x) = f(x) - \frac{f(y)}{P^*(y)} P^*(x).$$

By (5),

$$U_n(f_y, y) = U_n(f, y) - \frac{f(y)}{P^*(y)} U_n(P^*, y) \to 0, \tag{7}$$

uniformly in y. We have

$$\begin{aligned} |U_n(f,y) - f(y)| &\le \left| U_n(f,y) - \frac{f(y)}{P^*(y)} U_n(P^*, y) \right| + \left| \frac{f(y)}{P^*(y)} U_n(P^*, y) - f(y) \right| \\ &= \left| U_n(f,y) - \frac{f(y)}{P^*(y)} U_n(P^*, y) \right| + \frac{|f(y)|}{|P^*(y)|} \left| U_n(P^*, y) - P^*(y) \right| \\ &\le \left| U_n(f,y) - \frac{f(y)}{P^*(y)} U_n(P^*, y) \right| + \gamma \|f\| \left| U_n(P^*, y) - P^*(y) \right| \end{aligned}$$

Using (7) and the fact that $U_n(P^*, y) \to P^*(y)$, uniformly, the theorem follows. □

Applications

Bernstein polynomials

Consider the operator $B_n\colon C([0,1]) \to C([0,1])$ generating the Bernstein polynomial

$$f \mapsto B_n(f)(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad n = 1, 2, \ldots$$

Clearly each B_n is linear and positive. Also notice that $B_n(1) = 1$, $B_n(t)(x) = x$, and $B_n(t^2)(x) = x^2 + \frac{x(1-x)}{n}$. We can therefore use

$$P_y(x) = (x-y)^2 = x^2 - 2yx + y^2 := x^2 g_1(y) - 2x g_2(y) + g_3(y)$$

in Korovkin's theorem to conclude that $B_n(f) \to f$ uniformly for all $f \in C([0,1])$.

Trigonometric sums

Consider the Féjer maps $\sigma_n\colon C(\mathbb{T}) \to C(\mathbb{T})$ given by

$$f \mapsto \sigma_n(f)(x) = \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) F_n(x-t)\, dt, \qquad F_n(\alpha) = \frac{1}{2(n+1)} \frac{\sin^2 \frac{(n+1)\alpha}{2}}{\sin^2(\alpha/2)}.$$

Using certain well-known trigonometric identities, one can verify that

$$\sigma_n(f) = \frac{s_0(f) + s_1(f) + \cdots + s_n(f)}{n+1},$$

where $s_k(f)$ is the k'th partial sum of the Fourier series of f:

$$s_k(f) = \frac{a_0}{2} + \sum_{j=1}^{k} \big(a_j \cos jx + b_j \sin jx\big),$$

with

$$a_j := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt\, dt, \quad \text{and} \quad b_j := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt\, dt.$$

Notice that σ_n is linear and positive. One easily verifies that for $n \geq 1$; $\sigma_n(1) = 1$, $\sigma_n(\cos) = \frac{n}{n+1} \cos x$, and $\sigma_n(\sin) = \frac{n}{n+1} \sin x$. Thus, Korovkin's Theorem applies with

$$P_y(x) = 1 - \cos(x-y) = 1 - \cos(y)\cos(x) - \sin(y)\sin(x),$$

and $\sigma_n(f) \to f$ uniformly for every $f \in C(\mathbb{T})$.