Location

Department of Mathematical Sciences, Fredrik Bajersvej 7G, G5-109.
September 22 – 23.

Main Reference:

[HSD]: M.W. Hirsch, S. Smale, R.L. Devaney, *Dynamical Systems, Differential Equations & an Introduction to Chaos*, 2nd ed., Elsevier, 2004.

Supplementary reading

- J.H. Hubbard, B.H. West, *Differential Equations: A Dynamical Systems Approach. Higher-Dimensional Systems*, Texts in Applied Mathematics **18**, Springer-Verlag, 1995.
- from the internet: M. Grinfeld, Higher-dimensional Dynamical Systems

The exponential map for matrices and solutions to systems of linear differential equations

Lecture

Thursday, September 22, morning session

Aims and Content

A single linear equation $x' = \lambda x$ has $x(t) = C\exp(\lambda t), C \in \mathbf{R}$ as its general solution. By analogy, a linear system $X' = \mathbf{A}X$ should have a general solution of the form $X(t) = \exp(\mathbf{A}t)\mathbf{C}, \mathbf{C} \in \mathbf{R}^n$. This is in fact true – and proven in exactly the same way – as soon as one has a consistent definition of the exponential of a matrix $\mathbf{A}$. The exponential of a matrix is defined via the power series (Taylor series) of the exponential map; one needs to show that the latter always converges in the matrix domain. It is easy to calculate exponential maps for *semi-simple = diagonalisable* matrices and for *nilpotent* matrices (matrices $\mathbf{A}$ for which $\mathbf{A}^k = \mathbf{0}$ for some $k > 0$). The general calculation relies on the fact that every quadratic matrix $\mathbf{A}$ can be decomposed as $\mathbf{A} = \mathbf{S} + \mathbf{N} = \mathbf{N} + \mathbf{S}$ with $\mathbf{S}$ semi-simple and $\mathbf{N}$ nilpotent.[1]

Where do these nilpotent matrices come from? If a matrix $\mathbf{A}$ has multiple eigenvalues, then it is similar to

[1] This calculation is not covered in the textbook, but it will be explained in the lecture.

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the sum of a diagonal matrix D and of a matrix N the only non-zero entries of which are "just over" the diagonal. Such a matrix is clearly nilpotent!

References:

[HSD], ch. 6.3 – 6.4, pp. 119 – 130.

Exercises:

[HSD], ch, 6, pp. 136 – 137, exc. 6,8,9,10.

Non-autonomous linear systems. Harmonic Oscillators

Lecture

Thursday, September 22, afternoon session

Aims and Content

One or several harmonic oscillators are the guiding example in our further study of linear systems. We have previously looked at just one (undamped) harmonic oscillator with a corresponding (simple) phase plot consisting of concentric circles. A (pair) of undamped oscillators leads immediately to a 4D phase space; it turns out that the essence is to understand just the *angular* movement on a 2D *torus* (a ring surface; imagine a donut!) It turns out that the system has closed orbits if and only if the ratio of the frequencies of the two oscillators is *rational*.

If one applies an external force to a harmonic oscillator, the associated differential equation will still be linear, but it will no longer be homogeneous. A standard method for solving nonhomogeneous equations is the *method of undetermined coefficients*; its bias is that the result is in terms of an indefinite integral that often is difficult or impossible to calculate exactly. Things work out nicely though in case of a forced harmonic oscillator – with interesting results on the long time behaviour.

References:

[HSD], ch. 6.2, p. 114 – 119, ch. 6.5, pp. 130 – 135

Exercises:

[HSD], ch. 6, pp. 137 – 138, exc. 11,12(a),(c),(f),(j)

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Existence and Uniqueness Theorems for general Dynamical Systems

Friday, September 23, morning session

Aims and Content

We treat a general system of nonlinear differential equations. Unfortunately, it is rarely possible to find the explicit solution for any initial value problem of a nonlinear system. In fact there are dynamical systems which do not have any solution to a given initial value problem. Even if there is a solution, then it may tend to infinity in finite time. We provide examples of such systems during the lecture. We will see that the initial value problem

$$\dot{X} = F(X),\ X(t_0) = X_0,$$

where $X_0 \in \mathbf{R}^n$ and the vector field $F : \mathbf{R}^n \to \mathbf{R}^n$ is *continuously differentiable*, has a unique solution on an open interval containing t_0.

We pose a question: if we vary the initial conditions slightly, does the corresponding solution also vary only slightly? During the lecture we give an affirmative answer to this question. That is, we show that the solution of a nonlinear system depends *continuously* on the initial conditions.

References:

[HSD], ch. 7.1 – 7.3, pp. 139 – 149

Exercises:

[HSD], ch. 1, pp. 18, exc. 11, 12(a); ch. 7, pp. 156, exc. 1 (a),(b),(c),(d).

The Variational Equation and Linearization at stationary points

Friday, September 23, afternoon session

Aims and Content

To an autonomous system $\dot{X} = F(X)$ we associate a *linear approximation*

$$\dot{Y} = A(t)Y,$$

where $A(t) = DF(X(t))$ with $DF(X(t))$ denoting the Jacobian matrix of F at the point $X(t)$ (X(t) is a solution of the nonlinear system). The question we want to answer is how accurate this approximation is. In particular if the initial condition is such that X_0 is an equilibrium point ($X(t) = X_0$) then $A \equiv A(t) = DF(X_0)$

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and the linear counterpart is autonomous. Our goal is to see that the solutions of a nonlinear system near the origin resemble the solutions of a linear approximation $\dot{Y} = AY$ at – that we have studied previously – at least in many interesting cases.

References:

[HSD], ch. 7.4, pp. 149 – 153; ch. 8.1, pp. 159 – 165

Exercises:

[HSD], ch. 8, pp. 184/185, exc.1(ii),(iii),(iv) and exc. 2.

Evaluation, part 2

We ask you to write a report (in groups of two or three) on the following exercises and hand it in (by mail or e-mail) no later than Monday, October 3:

- from [HSD], pp. 136/137, exc. 7
- from [HSD], pp. 184/185, exc.1(i),(v)

Last day

Thursday, October 6. Equilibria in Nonlinear Systems, stability and bifurcations. [HSD], ch. 8.2 – 8.5.

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