Location: Nov. 28 in Kroghstræde 3 room 1115. Nov. 29. and 30. in Kroghstræde 7 room 63.

Main References

A T. Apostol, *Mathematical Analysis*, Addison-Wesley.

Differentiability.

Mon, 29.11, 9 – 12

Startup

Discussion of topics related to the 2. block:

- your questions related to the lectures
- problems with exercises

Lecture

Aims and Content

After having emphasized continuity, we now get to *differentiability*, a concept that cannot be defined on general topological spaces, but only on open subsets of Euclidean space (or spaces built out of those):

From the basic year and also from math in high school, you already know about many differentiable functions – and you can certainly calculate the derivative or the partial derivatives of many functions. You have seen a definition of differentiability of a function $f:\mathbb{R}\to\mathbb{R}$, and you have perhaps been working with the Jacobi matrix for functions $f:\mathbb{R}^n\to\mathbb{R}^m$. For a function like $f(x)=\sin(x)$, everything is clear, but what about

$$f(x)=\begin{cases} x\sin(1/x) & \text{for } x>0\\ 0 & \text{for } x\leq 0\end{cases}$$

For this function your definition of differentiability from high school will tell you what to do, but what about functions of more than one variable? The function

$$f(x,y)=\begin{cases} \frac{xy^2}{x^2+y^4} & \text{for } (x,y)\neq(0,0)\\ 0 & \text{for } (x,y)=(0,0)\end{cases}$$

has derivatives in all directions – in particular it has all its partial derivatives, but it is not even continuous at $(0,0)$! Differentiable functions ought to be continuous,

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at least, and thus the right definition requires more than the existence of partial and directional derivatives.

The aim of this part of the course is to give the right definition of differentiability. We will emphasize an interpretation via *approximations by linear maps* or *linear tangent spaces* to the graph of the function.

References

A Chapter 12.1-12.5 and 12.7-12.10

Exercises

1. In Apostol p. 345, the function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

is studied. Prove, or convince yourself, that you understand Apostols proof, that this function has all directional derivatives, and that it is not continuous at $(0,0)$.

A 12.4, 12.5, 12.7.

Complex differentiability. Higher order derivatives.

Mon, 28.11., 12:30 – 15

Lecture

Aims and Content

A *Complex* differentiable function

$$f(z) = u(x,y) + iv(x,y), \qquad z = x + iy,$$

enjoys especially nice properties. In particular, the real and imaginary parts have to satisfy the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The converse is also true in the sense that if u and v are differentiable at a point c and they satisfy the Cauchy-Riemann differential equations, then f is *complex* differentiable at c.

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Can one ensure differentiability of a function of several variables by inspections of its partial derivatives? Yes, there is a sufficient (but not necessary) condition, using the continuity of the partial derivatives. An important tool in the proof is the mean value theorem in one and in several variables, which is interesting for other purposes, as well.

References

A Sections 5.15-5.16, 12.11 – 12.13 & 12.6.

Exercises

1. Show: There is no C^2-function $f : \mathbf{R}^3 \to \mathbf{R}$ such that $\nabla f(x,y,z) = (y^2z, 2xyz, xy^2+y)$.
2. **A** 12.20 (interpret the result by a drawing); 12.27
3. **A** 5.36.
4. **A** 5.35.

The implicit and inverse function theorem and some consequences.

Tue, 29.11, 9:00-12:00 .

Lecture

Aims and Content

The differential Df_p of a function $f : \mathbb{R}^n \to \mathbb{R}^n$ at $p \in \mathbb{R}^n$ is a linear approximation to the function. If f is a *diffeomorphism*, the chain rule implies that $Df_p \circ D(f^{-1})_{f(p)} = D(f^{-1})_{f(p)} \circ Df_p = I$, the identity map. In particular, the differential Df_p is a *linear isomorphism*. This is not very surprising, but did you know that there is a (partial) converse? Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^l function ($l \geq 1$) such that Df_p is a linear isomorphism. Then there is a neighbourhood X of p such that $f : X \to f(X)$ is a bijection. Moreover, the inverse is a C^l function. This is the inverse function theorem. We will prove this theorem as an application of the Banach fixed point theorem, where the Mean Value Theorem is used to show that a certain map is a contraction. (Apostol's proof is a bit different.)

The implicit function theorem is a consequence of the inverse function theorem. It is a statement about the set of solutions of a system of differentiable equations: Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be a C^l function and let $a \in \mathbb{R}^n$ be a *regular* *value.* Then $f^{-1}(a) = \{x \in \mathbb{R}^{n+k} | f(x) = a\}$ – the set of solutions of the n equations in $(n+k)$ indeterminates given by f – is a k-dimensional C^l submanifold of $\mathbb{R}^{n+k}$. More precisely: For each point in the solution set, $p \in f^{-1}(a)$ there is a neighborhood V_p, open in $\mathbb{R}^{n+k}$ such that $V_p \cap f^{-1}(a)$ can be written as the graph of a C^l-function $g : U_p \to \mathbb{R}^n$, where U_p is an open subset of $\mathbb{R}^k$ and $V_p \cap f^{-1}(a) = \{g(y), y) | y \in U_p\}$

There are lots of applications of these theorems. Here are a few:

There is a bijection between the set of all nxn real matrices $M_n(\mathbb{R})$ and $\mathbb{R}^{n^2}$. The determinant function $Det : M_n(\mathbb{R}) \to \mathbb{R}$ is then a smooth function from $\mathbb{R}^{n^2}$ to $\mathbb{R}$ with 1 as a regular value. Hence, $SL(n) = \{A \in M_n(\mathbb{R}) | Det(A) = 1\}$ is a smooth manifold, i.e., it looks like a graph locally.

Another example: The set of *orthogonal* matrices $O(n)$ – important in classical physics and in robotics – is a smooth submanifold of $M_n(\mathbb{R})$. It is the kernel of the map $S : M_n(\mathbb{R}) \to Sym_n$, $S(A) = AA^T$, into the space of all *symmetric* $n \times n$ matrices, which we can think of as $\mathbb{R}^{\frac{n(n+1)}{2}}$.

References

- Serge Lang: Real Analysis, Addison Wesley,1973. I will send you this on Monday.

A 13.1-13.4 With the following corrections: On page 372 line 12, the section beginning with “Let B be an n-ball”... and ending below the figure with “and (d).” should be replaced by: Since $f_{|B(a)}$ is open, $Y = f(B(a))$ is open and $g = f^{-1}$ is continuous.

In the statement of the implicit function theorem, p.374, add condition d): There is an open set $X \subseteq S$, s.t. $(x,t) \in X \cap f^{-1}(0) \Rightarrow x = g(t)$ and s.t. $(g(t),t) \in X$ for all $t \in T_0$.

In the last line on p.374, replace $X = F^{-1}(Y)$ by $Y = F(X)$.

The very last lines of the proof (on p.375): Use the condition d) to prove uniqueness. The set X in the added condition d), is the set X in the proof.

Exercises

1. Let $F(x,y,z)$ be a C^1 function from $\mathbb{R}^3 \to \mathbb{R}$. At some points $p = (x_0, y_0, z_0)$, the equation $F(x,y,z) = a$ defines z as a C^1 function of (x,y) in a neighbourhood of p. Apply the implicit function theorem to state conditions. When these conditions hold – prove that $\frac{\partial z}{\partial x} = -\frac{D_x F}{D_z F}$ and give a formula for $\frac{\partial z}{\partial y}$ under the corresponding condition.

2. For which values of a is $\{(x,y,z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = a\}$ locally a graph? Draw the graph for $a = 1$ and $a = -1$. What “goes wrong” for $a = 0$?

Optimization: Lagrange[1] Multiplier and Kuhn-Tucker methods

Tue, 29.11., 12.30 – 15

Lecture

Aims and Content

Optimization of some function - the cost, the energy, ... with respect to boundary conditions is used in economics, in engineering etc. As a consequence of the implicit function theorem, we get the Lagrange method for optimization with side conditions. It provides necessary conditions for a point to be an extreme point for the restriction of a function f to a subset, if this subset S is defined in terms of some equations $g_i(x_1, x_2, \dots, x_n) = 0$.

In case the subset S is defined by equations *and inequalities*, the Kuhn-Tucker method yields necessary conditions for extreme points of f with respect to S.

References

A 13.5-13.7.

- http://are.berkeley.edu/courses/ARE211/currentYear/lecture_notes/mathNPP2-05.pdf

Exercises

A 13.12, 13.14.

1. Let $\mathbf{A}$ denote a *symmetric* $n \times n$ matrix. Prove that

$$\mu := \max\{\mathbf{x} \cdot \mathbf{A}x | \ \| \mathbf{x} \| = 1\}$$

is an eigenvalue of $\mathbf{A}$, i.e., there is a vector $\mathbf{y} \in \mathbf{R}^n$ such that $\mathbf{A}y = \mu\mathbf{y}$. In particular, every symmetric matrix has real eigenvalues.
(Hint: The Lagrange multiplier condition on the maximum of the function $f : \mathbf{R}^n \to \mathbf{R}^n$, $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A}\mathbf{x}$ under the side condition $\mathbf{x} \cdot \mathbf{x} = 1$. The symmetry of A yields a simple expression for Df.)

Banach fixed point theorem and differential equations

Wed, 30.11, 9:00-12:00 .

[1] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Lagrange.html

Lecture

Aims and Content

At first, it is not obvious that the Banach fixed point theorem can be used to study solutions of initial value problems of the type

$$\frac{dx}{dt} = f(t,x), \qquad x(t_0) = x_0, \tag{1}$$

where $f : U \to \mathbb{R}^n$ is a continuous function on an open subset U of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. To make the connection we rewrite the initial value problem as an equivalent integral equation,

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

A solution $x(t)$ of this integral equation is also a fixed point for the mapping

$$K(x)(t) := x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

which can be shown to be a contraction on a certain space of bounded continuous functions, provided f satisfies the Lipshitz condition

$$|f(t,x) - f(t,y)| \leq L|x-y|, \qquad (t,x),(t,y) \in U, \tag{2}$$

for some finite constant L. We can then apply the fixed point theorem to conclude that there exists a unique local solution of (1).

References

- Sections 2.1-2.3 of the lecture notes (243 pages, so you may not want to print the full document),

  `http://www.mat.univie.ac.at/~gerald/ftp/book-ode/ode.ps`

Exercises

- Suppose $G \subset \mathbb{R}^2$ is open and that $f : G \to \mathbb{R}$ is a continuous function having a continuous partial derivative f_y on G. Prove that $f(t,x)$ satisfies the Lipshitz condition (2) on any bounded open subset $G_1 \subset G$ satisfying $\overline{G_1} \subset G$.

- Which of the following functions $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies a Lipschitz condition on some open set containing the origin?

1. $f(t,x) = \frac{1}{1-x^2}$.
2. $f(t,x) = |\,x|^\alpha$ for $\alpha \in (0,1]$.
3. $f(t,x) = x^2 \sin(1/x)$.

- Consider the initial value problem $\frac{dx}{dt} = x$ with $x(0) = 1$. Let $x_0 = 1$ and calculate $x_n(t) = K^n(x_0)(t)$. Evaluate $\lim_{n\to\infty} x_n(t)$. Do you get the expected result? This type of iteration process is called *Picard* iteration.

More on differential equations

Wed, 30.11, 12:30-15:00.

Lecture

Aims and Content

We study the initial value problem

$$\frac{dx}{dt} = f(t,x), \qquad x(t_0) = x_0,$$

in more detail. We will use Gronwall's inequality to study the dependence of solutions on the initial conditions.

We conclude by considering examples that show why one needs to impose a Lipschitz condition on $f(t,x)$ to ensure convergence of the Picard iteration and to insure local uniqueness of the solution.

If time permits, solutions to selected exercises will be given.

References

- Section 2.3 of the lecture notes

  `http://www.mat.univie.ac.at/~gerald/ftp/book-ode/ode.ps`

Exercises

- Consider the function

$$\phi(t,x) = \begin{cases} 0, & \text{for } t \le 0 \\ 2t & \text{for } t > 0 \text{ and } x \le 0 \\ 2t - \frac{4x}{t} & \text{for } t > 0 \text{ and } 0 < x < t^2 \\ -2t & \text{for } t > 0 \text{ and } x \ge t^2. \end{cases}$$

  Convince yourself that $\phi(t,x)$ is continuous, and show that ϕ does not satisfy a Lipschitz condition on any open set containing the origin. Can you find (or perhaps guess) a solution of the IVP: $x'(t) = \phi(t,x)$, $x(0) = 0$?

- Find a solution of the IVP,

$$\frac{dy_1}{dx} = y_2, \qquad y_1(0) = 1$$
$$\frac{dy_2}{dx} = -y_1, \qquad y_2(0) = 0.$$

- Rewrite each second order differential equation

$$a) \quad \frac{d^2y}{dx^2} = y, \qquad b) \quad \frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = \cos(x), \qquad c) \quad \frac{d^2y}{dx^2} = F(x, y, y').$$

  as a system of first order differential equations. Hint: Let $y_1(x) = y(x)$ and $y_2(x) = \frac{dy}{dx}$.

- Old problems.

Course Evaluation

The present course

Please, give us your comments and proposals (for another try) about

- the form of the course (lectures, exercises etc.)
- the literature
- the accessibility and the workload
- the relevance
- etc

Demands for other math courses at the Ph.D.-level

All sorts of math courses that you have always dreamt about but never dared to ask for...

Bye-bye

Finally, we would like to thank you for your kind and active participation.

Best regards,

Lisbeth and Morten.

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