Main References

A T. Apostol, *Mathematical Analysis*, Addison-Wesley.

DS D. A. Santos, *Linear Algebra Notes*, electronically available at `http://www.openmathtext.org/lecture_notes/new_linearalgebra.pdf`. We will *only* use Chapters 5 & 6 (pages 100-132).

Overall structure

We plan to organize the sessions to include both lectures and hands-on exercise sessions with a daily scheme like

9	– 10	Lecture 1
10	– 11	Exercise session 1
11	– 12	Lecture 2
12	– 12:30	Lunch break
12:30	– 13:20	Lecture 3
13:20	– 14:10	Exercise session 2
14:10	– 15	Lecture 4

Introduction and Metric Spaces

Mon, 24.10., 9 – 12, Kroghstæde 7 room 63

Startup

Welcome and presentation of the lecturers and the participants. Discussion of expectations with the course and its form. In particular: preparation, work load, role of exercises and evaluation.

We also plan to show a few examples intended to demonstrate that it can be helpful to study problems within a more rigorous mathematical framework.

Lectures

Aims and Content

Metric spaces yield a quite general framework in which one studies properties which arise from a distance function on a set. Once such a distance function is defined and satisfies some very basic axioms, one can define continuous functions, bounded functions, convergent sequences and many other concepts which you may

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already know from Euclidean space $\mathbb{R}^n$. The most important example of a metric on $\mathbb{R}^n$, which we will study in some detail, is the Euclidean metric given by

$$d(\mathbf{x},\mathbf{y}) := \Big(\sum_{j=1}^{n}(x_j-y_j)^2\Big)^{1/2}, \qquad \mathbf{x},\mathbf{y}\in\mathbb{R}^n.$$

In the first session will give examples of metric spaces, and define open/closed subsets of metric spaces.

A striking example of the power of this kind of abstraction will be the application of the Banach fixed point theorem to prove existence and uniqueness of local solutions to ordinary differential equations. We will study the Banach fixed point theorem and its applications in the 2nd block.

Later in block 2, we will see that metric spaces provide important examples of so-called topological spaces, and many concepts can be defined in that more general setting.

References

[**A**] 3.2 (also Theorem 1.23 on page 14), 3.13 and 3.14 to p.62. Please, have a look at these sections prior to the lecture.

Exercises

1. In **A** p. 61 Examples 1, 3, 8 and 9 define different metrics on $\mathbb{R}^2$ (and other sets, but in particular on $\mathbb{R}^2$). Draw examples of balls, $B_d(x,r)$ in each of these metrics.
2. Let (M,d) be a metric space. Prove that $B_d(x,r)$ is open for all $x\in M$ and $r>0$.
3. **A**, Exc. 4.66(a), p. 102. This example will be used extensively later. Note: the supremum of a point set is discussed in **A**, §1.10.
4. Verify that the *discrete metric* defined in Example 3, p. 61 in **A**, is a metric.
5. **A**, Exc. 3.31, p. 67.

Convergence, compact sets, Cauchy sequences

Mon, 24.10., 12:30 – 15, Kroghstræde 7, room 63

Lectures

Aims and Content

Many of the concepts which you probably know from $\mathbb{R}^n$, e.g. *convergence* of a sequence, are easily generalized to metric spaces in general. Such a generalization allows one to consider convergence of a sequence of functions in a space of functions with a metric. It is an important point, that a given sequence may converge in one metric and not converge in another. To see that a sequence converges, one needs to know the limit point, but in some cases, it may be enough to know that the sequence is a *Cauchy-sequence*.

In general, a metric space may be quite strange - sequences, which “ought to converge” - Cauchy sequences - may not converge; a continuous map between metric spaces $f : M_1 \to M_2$, may be a bijection without the inverse being continuous. *Compact* subsets of metric spaces are subsets with better properties: A Cauchy sequence in a compact subset will converge, a continuous bijection $f : M_1 \to M_2$ where M_1 is compact, will in fact have continuous inverse.

References

[A] 3.14 from p. 62, 3.15, 3.16, 4.1, 4.2, 4.3, - Please have a look at this before the lectures. Do not get lost in the proofs – they are not easily digested and may be easier to grasp after the lectures

Exercises

1. In Example 1 and 8 p. 61, different metrics are defined on $\mathbb{R}^n$. Prove that a subset $A \subseteq \mathbb{R}^n$ is open with respect to the metric in Ex.1 if and only if it is open w.r.t. the metric defined in 8.

2. A 3.32, 3.39, 3.40

Complete metric spaces and a closer look at continuous functions.

Tue, 25.10., 9 – 12, Kroghstræde 7, room 63

Startup

Discussion of the topics of Monday's lectures and exercises.

Lectures

Aims and Content

A metric space in which every Cauchy sequence converges, is *complete*. Euclidean space, $\mathbb{R}^n$ is in fact complete. Another important example of a complete metric space is the space of bounded real functions $B(S, \mathbb{R})$, where T is some set and the metric is the supremum metric as defined in the exercises Monday – **A**, Exc. 4.66(a), p. 102. We will study this example in detail.

Hence, in order to make convergence in a metric space X work, one has to make sure that all Cauchy sequences converge within X. If a given space is not complete one may *construct* a completion. The real numbers $\mathbb{R}$ represent the completion of the rational numbers $\mathbb{Q}$. Completions of spaces of continuous functions may include non-continuous functions (like the Dirac-δ function, that make perfectly sense!)

Continuous functions are defined in terms of the metric, but in fact, continuity may be defined in terms of the open sets of the space; and as we saw in exercise 1 on Monday afternoon, different metrics may give rise to the same open sets. This observation will lead to the introduction of topological spaces in the next block.

References

A 4.4, 4.5, 4.8, 4.9, 4.11, 4.12, 4.13, 4.14.

Exercises:

1. A 4.7 and 4.8
2. Consider the metric subspace $M = \mathbb{R} \setminus \{0\}$ of $\mathbb{R}$. Prove that the sequence $\{\frac{1}{n}\}_{n \in \mathbb{N}}$ is a Cauchy sequence in M and that it does not converge in M.
3. On p.219 Ex. 1. Prove that the function f is in fact the limit of the functions f_n.

Vector Spaces

Tue, 25/10., 12:30 – 15, Kroghstræde 7, room 63

Lecture

Aims and Content

In a general metric space (M, d) no meaning is attached to any form of sum of elements of M. However, in a vector space we may add elements together and we may also multiply elements by a scalar. In this session we will study how to combine the properties of metric spaces with the added algebraic structure provided by the vector space. You have already met interesting examples of vector spaces so far, such as $\mathbb{R}^n$ and the family of bounded real functions $B(S, \mathbb{R})$.

Normed vector spaces are of particular interest to us since such spaces are also metric spaces. A normed vector space is a vector space X together with a mapping $\|\cdot\| : X \to [0, \infty[$ with the properties

1. $\|x\| = 0$ if and only if $x = 0$,
2. $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and every scalar α,
3. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$ (triangle inequality).

Any normed vector space X can be turned into a metric space by introducing the metric $d(x, y) = \|x - y\|$ for $x, y \in X$. Hence, given two normed vector spaces X, Y we may consider continuous maps $f : X \to Y$. The class of continuous *linear* maps between X and Y is particularly important.

References

DS §5.1-§5.5 & §6.1-§6.2.

Exercises

1. Show that the set of polynomials $\mathcal{P}$ with coefficients in $\mathbb{R}$ is a vector space over $\mathbb{R}$. For $n \in \mathbb{N}$, is the set $\widetilde{\mathcal{P}_n} = \{p \in \mathcal{P} : \deg(p) = n\} \cup \{0\}$ a subspace of $\mathcal{P}$? What about $\mathcal{P}_n = \{p \in \mathcal{P} : \deg(p) \leq n\}$?
2. Let $\mathbb{R}^2$ have the usual scalar multiplication, but let addition $\boxplus$ be defined on $\mathbb{R}^2$ by
$$(x, y) \boxplus (r, s) := (x + r, 2y + s).$$
Determine whether $\mathbb{R}^2$ with these operations is a vector space.
3. Is $\mathbb{C}$ a vector space over $\mathbb{R}$? Also, is $\mathbb{R}$ a vector space over $\mathbb{C}$?
4. Let $T : V \to W$ be a linear map between finite dimensional vector spaces. Show that T is continuous.
5. Let $g \in B(S, \mathbb{R})$. Define $T : B(S, \mathbb{R}) \to B(S, \mathbb{R})$ by $Tf(x) = g(x)f(x), \forall x \in S$. Verify that T is linear. Is T continuous?

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Evaluation

We ask you to work on the following exercise in groups of 2 or 3 and to hand in a solution. Deadline for this is Wednesday 2. November

Let (M, d) be a metric space.

1. Prove that M and the empty set $\emptyset$ are open sets.

2. Given a finite collection of open subsets $U_1, \ldots, U_n$ of M, prove that the intersection $\bigcap_{i=1}^{n} U_i$ is open.

3. Given an arbitrary collection of open subsets of M, $\{U_i | i \in I\}$, where I is some index set (not necessarily finite). Prove that the union $\bigcup_{i \in I} U_i$ is an open set.

4. Let n be an integer and let $]-1/n, 1/n[$ be the open interval in $\mathbb{R}$. What is the intersection $\bigcap_{n=1}^{\infty}]-1/n, 1/n[$.

Plan for the 2. block

Date: 7-8/11.2005 in Kroghstræde 7, room 63.

- Topological spaces
- Uniform convergence
- The fixed point theorem
- Stone-Weierstrass approximation
- Other approximation results.

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