Location

Kroghstræde 7, room 63.

Main References

A T. Apostol, *Mathematical Analysis*, Addison-Wesley.

BV M. Bökstedt and H. Vosegaard, *Notes on point-set topology*, electronically available at `http://home.imf.au.dk/marcel/gentop/index.html`.

Examples of metric spaces. Uniform Convergence

Mon. 7/11, 9 – 12

Startup

Discussion of topics related to the 1. block:

- your questions related to the lectures
- problems with exercises

Lectures

Aims and Content

Some of you have asked where metric spaces and convergence etc. come up in engineering sciences. Convergence properties are actually used in most of the tools which use a computer program to solve a (big enough) problem. Many such tools do, what computers are really good at: The same thing over and over again. Often, an initial guess of a solution (a constant function for instance) is fed into a loop, where, hopefully, the outcome is closer to the real solution than the first guess. This outcome is then fed back to the loop etc. until the program stops - perhaps because the output of the loop resembles the input, and one concludes that it is close to the true solution.

An example of a widely used metric is the Hamming distance between strings (binary strings, words, ...) The distance is the number of places, where the strings differ. This is used in for instance coding theory, and also for some optimization algorithms. Other distances between strings are used: The Levenshtein metric measures the number of edit operations, it takes to get from one string to the other - editing is insertion, deletion and substitution. Various metrics between strings are used in search algorithms.

In image compression, the compression is designed in such a way that the image after decompression is within a small enough ball around the original image - so that the human eye cannot tell the difference. This requires a mtric on images.

In control theory for a robot arm,the arm is restricted to move not in $\mathbb{R}^3$ but on another geometric object - depending on the number of joints on the arm. Control theory then uses a metric on the geometric object to decide stability.

Moreover, metric spaces provide a nice example of a mathematical area, where one can get quite far from a few simple definitions - this should allow you to focus on the logical arguments and hence you should be able to learn something about how such arguments should proceed - even in more complicated situations such as your own projects.

Uniform Convergence Different metrics give rise to different convergence criteria. A quite natural criterion for convergence for a sequence of real functions $\{f_k : S \to T\}_{k\in\mathbb{N}}$ may be, that each sequence of values $f_k(x)$ converge, i.e., that they converge pointwise. However, a sequence of continuous real functions $f_k : I \to \mathbb{R}$, may converge pointwise to a function f, $f_n(x) \to f(x)$ for all $x \in I$, without the limit function being continuous. If, however, a sequence of functions converge *uniformly*, then the limit function is continuous. For bounded functions, $f_n \in B(I, \mathbb{R})$, uniform convergence means convergence in the supremum metric.

We will see in the exercises, that if an *increasing* sequence of functions $f_n : K \to \mathbb{R}$ on a compact K converges pointwise, then it converges uniformly. This fact (Dini's[1] theorem) is very useful in integration theory.

References

[A] 9.1-9.5

Exercises

A 9.5

- This exercise takes you through a proof of Dini's theorem: Let $f_n : K \to \mathbb{R}$ $n \in \mathbb{N}$ be continuous functions on a compact set K and suppose that the sequence is increasing: $f_k(x) \leq f_{k+1}(x)$ for all $x \in K$ and all $k \in \mathbb{N}$. If $f_n(x) \to f(x)$ for all $x \in K$ and if f is continuous, then $f_n \to f$ uniformly.

  1. Let $g_n = f - f_n$. Prove that $g_n \geq g_{n+1} \geq 0$ for all $n \in \mathbb{N}$ and that $g_n(x) \to 0$ for al $x \in K$
  2. Let $\epsilon > 0$. Prove that for any $x \in K$ there is an N_x such that $g_n(x) < \epsilon/2$ whenever $n \geq N_x$.

[1] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Dini.html

3. Prove that there is a $\delta_x > 0$ such that
$$y \in B(x, \delta_x) \Rightarrow |g_{N_x}(y) - g_{N_x}(x)| < \epsilon/2$$
and consequently (why?)
$$n \geq N_x, y \in B(x, \delta_x) \Rightarrow g_n(y) < \epsilon$$
4. Let $\mathcal{U} = \bigcup_{x \in K} B(x, \delta_x)$. Use compactness to refine this cover of K to get $B(x_1, \delta_{x_1}), B(x_2, \delta_{x_2}), \ldots, B(x_k, \delta_{x_k})$.
Let $N = max\{N_{x_i}, i = 1, \ldots, k\}$ and conclude:
For $n \geq N$ $g_n(y) < \epsilon$ for all $y \in K$.
5. Now you proved the theorem – can you see that?

Topological Spaces

Mon, 7/11, 12.30 – 15

Lectures

Aims and Content

Topological spaces are more general and more flexible than metric spaces; nevertheless, they share many common features with the latter. Surprisingly, it is sometimes easier to show certain properties within the topological rather than in the metric framework. The *topology* of a space X is defined as the set of open subsets of X which has to be closed under unions and finite intersections. The same underlying space can thus have many different topologies most of them with no interest at all. Metric spaces come equipped with a natural metric topology; note that different metrics can give rise to the same topology.

Some basic vocabulary concerning topological spaces will be explained and has to be digested: closed sets, interior points of a subset; the interior $Y^o \subset Y$ and the closure $\bar{Y} \supset Y$. As for metric spaces, we use continuous maps to relate two topological spaces. The definition (Def. 2.2.1 in [BV]) is not easy to understand rightaway; it uses the concept pre-image of a subset, defined as follows: For a map $f : X \to Y$ and a subset $V \subset Y$, let
$$f^{-1}(V) := \{x \in X : |f(x) \in V\} \subset X.$$
Note that this definition does not require the original map to have an inverse! Other phrasings of the definition (all of them useful) are given in Prop. 2.2.3/4. Note the ease with which one can show that the composition of two (or more) continuous maps is continuous (Prop. 2.2.6)! We will try to complement the lecture notes with a range of examples as well.

References

[BV] Ch. 2.1 & 2.2, pp. 17–23. Please, have a look at these sections prior to the lecture.

Exercises

- Verify carefully that a metric spaces is a topological space. Hint: Use hand-in from last time.
- Let $X = \{a, b, c, d\}$,
$$\mathcal{F}_1 = \{\emptyset, \{a\}, \{b\}, \{a,b\}, \{a,b,c\}, \{a,b,d\}, X\}$$
$$\mathcal{F}_2 = \{\emptyset, \{a\}, \{a,b\}, \{a,c\}, \{a,b,c\}, X\}.$$
Verify that $\mathcal{F}_1$ and $\mathcal{F}_2$ are topologies on X.
- Let $\mathcal{F}_1$ and $\mathcal{F}_2$ be topologies on a set X. Show that the identify map $I : (X, \mathcal{F}_1) \to (X, \mathcal{F}_2)$ is continuous if and only if $\mathcal{F}_2 \subseteq \mathcal{F}_1$ (we say, $\mathcal{F}_1$ is *stronger* than $\mathcal{F}_2$ when $\mathcal{F}_2 \subseteq \mathcal{F}_1$).
- **BV** Ex. 2.1.15.

The fixed point theorem

Tue, 8.11., 9 – 12

Lectures

Aims and Content

You may recall Newton's method to find a root of a (sufficiently "nice") function $f : [a,b] \to \mathbf{R}$. We make an initial guess $x_0 \in [a,b]$ (preferably close to the root) and then we find the root as the limit of the recurrence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The success of this and many other iteration procedures can be made sure if the requirements of the Banach fixed point or contraction theorem are satisfied. A fixed point for a map $f : X \to X$ is a point $p \in X$ satisfying $f(p) = p$. The proof of the Banach fixed point is surprisingly simple, its applications vast. A contraction is a map $f : X \to X$, with X a complete metric space, such that for some constant $0 < s < 1$

$$d(f(x), f(y)) \leq sd(x,y) \qquad \text{for all} \quad x, y \in X.$$

Theorem. Any contraction has a unique fixed point in X. This fixed point is the limit of the sequence x_0, $f(x_0)$, $f(f(x_0))$,... for every choice of $x_0 \in X$.

Examples of applications are the proof for the existence and uniqueness of solutions of ordinary differential equations (and the construction of the solution by the Picard-Lindelöf method), and the construction of fractals from iterated function systems. In this session we will study Newton's method in some detail. In the 3rd block, we will use the fixed-point theorem to study existence and uniqueness of solutions of ordinary differential equations.

References

[A] 4.21.

Exercises

1. Find the fixed point(s) of $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^2 - 4x + 4$. What about $q(x) = x^2 - 2x + 4$?
2. A, ex. 4.69.
3. Let X be a complete metric space, and let $f : X \to X$. Suppose $g = f \circ f$ is a contraction on X. Show that f has a unique fixed point.
4. A, ex. 4.71.(a).

Approximation: The Stone-Weierstrass Theorem

Tue, 9.11., 12.30–15.00

Lecture

Aims and Content

General continuous functions are often difficult to describe and to handle. Is it always possible to approximate them by simpler functions, e.g., by polynomials? Certainly not always, since a polynomial tends to either ∞ or $-\infty$ for numerically large values. Hence, polynomials cannot approximate bounded functions (like sin) on the reals.

But a famous theorem of Karl Weierstrass[2] tells us, that every continuous function on a *compact* interval can be *uniformly* approximated by polynomials, cf. Apostol, Thm. 11.17. We want to present a slick proof for this fact, that allows many generalisations. In this version, it goes back to Marshall Stone[3], and it is

[2] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Weierstrass.html
[3] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Stone.html

called the Stone-Weierstrass theorem: Certain *algebras* of continuous functions on a compact set are *dense* in the space of all continuous functions on this set wrt. the sup-norm. For the proof, one has essentially to show that the function $f(t) = |t|$ can be arbitrarily well approximated by polynomials on the interval $[-1, 1]$. The rest of the proof is almost algebraic in nature.

As one of the applications, one obtains that every periodic continuous function can be uniformly approximated by trigonometric functions (Fourier polynomials). This result is first of all abstract and does not provide you with a method to detect a sequence of polynomials converging uniformly to a given continuous functions. There are concrete algorithms, e.g., using Bernstein[4] polynomials as approximations of continuous functions on an interval.

References

1. **A** Sect. 11.15, p. 322 (using a result of Fejér)
2. S. Lang, *Real Analysis*, Addison-Wesley, Sect. III.1, pp. 47 – 51. We will send this note to you.
3. W. Rudin, *Principles of Mathematical Analysis*, McGraw-Hill, pp. 159 – 165
4. *More about the Stone-Weierstrass theorem than you probably wanted to know*, `www.math.rutgers.edu/courses/501/501-f99/stnw_502.pdf`

Exercises

1. Apostol p. 102 ex. 4.67. Make sure, you understand the statement in exercise 4.66
2. exercises from earlier that you did not finish yet.

Evaluation

Please work in groups of two or three on the exercise 4.68 in Apostol p. 102 and hand in a solution no later than Monday, November 21.

Plan for the 3. block

Date: 28./30.11.2005

- Differentiability for functions defined on opens subsets of $\mathbf{R}^n$

[4] `http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Bernstein_Sergi.html`

- The differential
- Linear approximation
- Taylor's formula for multivariate functions
- Differentiability of complex functions
- The inverse and implicit function theorem and some consequences
- Lagrange and Kuhn Tucker optimization.
- Existence and uniqueness of solutions of ordinary differential equations.

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