Location: Fredrik Bajers Vej 7G, room G5-109.

Main References

W W.R.Wade, An introduction to Analysis, Fourth edition, Pearson.

The Banach fixed point theorem

Mon, 23.11., 9 – 12

Lectures

Aims and Content

You may recall Newton's method to find a root of a (sufficiently "nice") function $f : [a, b] \to \mathbf{R}$. We make an initial guess $x_0 \in [a, b]$ (preferably close to the root) and then we find the root as the limit of the recurrence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

The success of this and many other iteration procedures can be made certain if the requirements of the Banach fixed point (also called the contraction) theorem are satisfied. A fixed point for a map $f : X \to X$ is a point $p \in X$ satisfying $f(p) = p$. The proof of the Banach fixed point is surprisingly simple, its applications vast. A contraction is a map $f : X \to X$, with X a complete metric space, such that for some constant $0 < s < 1$

$$d(f(x), f(y)) \leq sd(x, y) \qquad \text{for all} \quad x, y \in X.$$

Theorem. Any contraction on a *complete* metric space has a unique fixed point in X. This fixed point is the limit of the sequence x_0, $f(x_0)$, $f(f(x_0))$,... for every choice of $x_0 \in X$.

Examples of applications are the proof for the existence and uniqueness of solutions of ordinary differential equations (and the construction of the solution by the Picard-Lindelöf method), and the construction of fractals from iterated function systems. In this session we will study Newton's method in some detail. In the afternoon session, we will use the fixed-point theorem to study existence and uniqueness of solutions of ordinary differential equations.

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References

M. Nielsen: A note on Banach's fixed-point theorem. Available for download from the course homepage.

Exercises

1. Find the fixed point(s) of $p : \mathbb{R} \to \mathbb{R}$ defined by $p(x) = x^2 - 4x + 4$. What about $q(x) = x^2 - 2x + 4$?
2. Show by counterexamples that the fixed-point theorem for contractions need not hold if either (a) the underlying space is not complete, or (b) the contraction constant $\alpha \geq 1$.
3. Let X be a complete metric space, and let $f : X \to X$. Suppose $g = f \circ f$ is a contraction on X. Show that f has a unique fixed point.
4. Let $X \to X$ be a function from a metric space (X, ρ) into itself such that
$$\rho(f(x), f(y)) < \rho(x, y), \qquad x, y \in X, x \neq y.$$
Show that f has at most one fixed-point, and give an example of such an f with no fixed-point.

The Banach fixed point theorem and differential equations

Mon, 23.11, 12:30–16:00 .

Lecture

Aims and Content

At first, it is not obvious that the Banach fixed point theorem can be used to study solutions of initial value problems of the type

$$\frac{dx}{dt} = f(t, x), \qquad x(t_0) = x_0, \tag{1}$$

where $f : U \to \mathbb{R}^n$ is a continuous function on an open subset U of $\mathbb{R}^{n+1}$, and $(t_0, x_0) \in U$. To make the connection we rewrite the initial value problem as an equivalent integral equation,

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\, ds.$$

A solution $x(t)$ of this integral equation is also a fixed point for the mapping

$$K(x)(t) := x_0 + \int_{t_0}^{t} f(s, x(s))\, ds,$$

which can be shown to be a contraction on a certain space of bounded continuous functions, provided f satisfies the Lipshitz condition

$$|f(t,x) - f(t,y)| \leq L|x-y|, \qquad (t,x),(t,y) \in U, \tag{2}$$

for some finite constant L. We can then apply the fixed point theorem to conclude that there exists a unique local solution of (1).
There are many other examples of fixed point theorems: The Brouwer fixed point theorem says that a continuous maps from a closed ball in Euclidean space to itself has at least one fixed point. This is used to proove existence of equilibria in game theory. Fixed point theorems for partially ordered sets (lattices in fact) are used in theoretical computer science.

References

- Sections 2.1-2.2 of the lecture notes (243 pages, so you may not want to print the full document),

  `http://www.mat.univie.ac.at/~gerald/ftp/book-ode/index.html`

Exercises

- Which of the following functions $f : \mathbb{R}^2 \to R$ satisfies a Lipschitz condition on some open set containing the origin?

  1. $f(t,x) = \frac{1}{1-x^2}$.
  2. $f(t,x) = |\,x|^\alpha$ for $\alpha \in (0,1]$.
  3. $f(t,x) = x^2 \sin(1/x)$.

- Consider the initial value problem $\frac{dx}{dt} = x$ with $x(0) = 1$. Let $x_0 = 1$ and calculate $x_n(t) = K^n(x_0)(t)$. Evaluate $\lim_{n\to\infty} x_n(t)$. Do you get the expected result? This type of iteration process is called *Picard* iteration.

Differentiability.

Wednesday, 25.11, 9 – 12

Startup

Discussion of topics related to Monday:

- your questions related to the lectures
- problems with exercises

Lecture

Aims and Content

After having emphasized continuity, we now get to *differentiability*, a concept that cannot be defined on general topological spaces, or on metric spaces. One needs more structure. The Frechet derivative, which is used in calculus of variations, is defined for functions on open subset of a *Banach space*, but we will restrict our attention to open subsets of Euclidean space (or spaces built out of those):

From the "Basis" year and also from math in high school, you already know about many differentiable functions – and you can certainly calculate the derivative or the partial derivatives of many functions. You have seen a definition of differentiability of a function $f : \mathbb{R} \to \mathbb{R}$, and you have perhaps been working with the Jacobi matrix for functions $f : \mathbb{R}^n \to \mathbb{R}^m$. For a function like $f(x) = \sin(x)$, everything is clear, but what about

$$f(x) = \begin{cases} x\sin(1/x) & \text{for } x > 0 \\ 0 & \text{for } x \leq 0 \end{cases}$$

For this function your definition of differentiability from high school will tell you what to do, but what about functions of more than one variable? The function

$$f(x,y) = \begin{cases} \frac{xy^2}{x^2+y^4} & \text{for } (x,y) \neq (0,0) \\ 0 & \text{for } (x,y) = (0,0) \end{cases}$$

has derivatives in all directions – in particular it has all its partial derivatives, but it is not even continuous at $(0,0)$! Differentiable functions ought to be continuous, at least, and thus the right definition requires more than the existence of partial and directional derivatives.

The aim of this part of the course is to give the right definition of differentiability. We will emphasize an interpretation via *approximations by linear maps* or *linear tangent spaces* to the graph of the function.

References

W Chapter 11.1-11.4, not the part on partial integration

Exercises

W 11.2.1, 11.2.8, 11.2.10, 11.2.9

Complex differentiability. Higher order derivatives.

Wed, 25.11., 12:30 – 16

Lecture

Aims and Content

A *Complex* differentiable function

$$f(z) = u(x,y) + iv(x,y), \qquad z = x + iy,$$

enjoys especially nice properties. In particular, the real and imaginary parts have to satisfy the Cauchy-Riemann differential equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The converse is also true in the sense that if u and v are differentiable at a point c and they satisfy the Cauchy-Riemann differential equations, then f is *complex* differentiable at c.

The partial derivatives of a function may be differentiable and thus give rise to higher order derivatives. Under which conditions is the result independent of the *order* of differentiation? And how can one use the 1st and higher order derivatives for *approximation* purposes? The answer is given by *Taylor*[1]'s formula for (approximation by a *multivariate polynomial* of a given degree) and an estimation of the remainder term. An important tool in the proof is the mean value theorem in one and in several variables, which is interesting for other purposes, as well.

References

- Apostol: Mathematical Analysis (Sections 5.15-5.16 & 12.6). A scanned copy will be sent to you by email.
- **W**, Section 11.5.

[1] http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Taylor.html

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Exercises

1. Show: There is no C^2-function $f : \mathbf{R}^3 \to \mathbf{R}$ such that $\nabla f(x,y,z) = (y^2 z, 2xyz, xy^2 + y)$.
2. Apostol (see the scanned notes page 126): Exercise 5.36.
3. Apostol: Exercise 5.35.

The implicit and inverse function theorem and some consequences.

Fri, 27.11, 9:00-12:00 .

Lecture

Aims and Content

The differential Df_p of a function $f : \mathbb{R}^n \to \mathbb{R}^n$ at $p \in \mathbb{R}^n$ is a linear approximation to the function. If f is a *diffeomorphism*, the chain rule implies that $Df_p \circ D(f^{-1})_{f(p)} = D(f^{-1})_{f(p)} \circ Df_p = I$, the identity map. In particular, the differential Df_p is a *linear isomorphism*. This is not very surprising, but did you know that there is a (partial) converse? Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a C^l function ($l \geq 1$) such that Df_p is a linear isomorphism. Then there is a neighbourhood X of p such that $f : X \to f(X)$ is a bijection. Moreover, the inverse is a C^l function. This is the inverse function theorem. We will sketch a proof of this theorem as an application of the Banach fixed point theorem, where the Mean Value Theorem is used to show that a certain map is a contraction. (Wade's proof is a bit different.)

The implicit function theorem is a consequence of the inverse function theorem. It is a statement about the set of solutions of a system of differentiable equations: Let $f : \mathbb{R}^{n+k} \to \mathbb{R}^n$ be a C^l function and let $a \in \mathbb{R}^n$ be a *regular value*. Then $f^{-1}(a) = \{x \in \mathbb{R}^{n+k} | f(x) = a\}$ – the set of solutions of the n equations in $(n+k)$ indeterminates given by f – is a k-dimensional C^l submanifold of $\mathbb{R}^{n+k}$. More precisely: For each point in the solution set, $p \in f^{-1}(a)$ there is a neighborhood V_p, open in $\mathbb{R}^{n+k}$ such that $V_p \cap f^{-1}(a)$ can be written as the graph of a C^l-function $g : U_p \to \mathbb{R}^n$, where U_p is an open subset of $\mathbb{R}^k$ and $V_p \cap f^{-1}(a) = \{g(y), y) | y \in U_p\}$

There are lots of applications of these theorems. Here are a few:

There is a bijection between the set of all nxn real matrices $M_n(\mathbb{R})$ and $\mathbb{R}^{n^2}$. The determinant function $Det : M_n(\mathbb{R}) \to \mathbb{R}$ is then a smooth function from

$\mathbb{R}^{n^2}$ to $\mathbb{R}$ with 1 as a regular value. Hence, $SL(n) = \{A \in M_n(\mathbb{R}) | Det(A) = 1\}$ is a smooth manifold, i.e., it looks like a graph locally.

Another example: The set of *orthogonal* matrices $O(n)$ – important in classical physics and in robotics – is a smooth submanifold of $M_n(\mathbb{R})$. It is the kernel of the map $S : M_n(\mathbb{R}) \to Sym_n$, $S(A) = AA^T$, into the space of all *symmetric* $n \times n$ matrices, which we can think of as $\mathbb{R}^{\frac{n(n+1)}{2}}$.

References

- Serge Lang: Real Analysis, Addison Wesley,1973. I will send you this.

W The Inverse and implicit function theorems are in 11.6. I will stress the examples.

The statement of the Implicit Function theorem is not very precise in Wade. I prefer (on p.431) to say. There is an open set $U \subset V$ with $(x_0, t_0) \in U$ an open $W \subset \mathbb{R}^p$ and a continuously differentiable function $g : W \to \mathbb{R}^n$ such that $F^{-1}(0) \cap U = \{(g(t), t) | t \in W\}$. In other words, $F^{-1}(0) \cap U$ is the graph of g.

Exercises

1. Let $F(x, y, z)$ be a C^1 function from $\mathbb{R}^3 \to \mathbb{R}$. At some points $p = (x_0, y_0, z_0)$, the equation $F(x, y, z) = a$ defines z as a C^1 function of (x, y) in a neighbourhood of p. Apply the implicit function theorem to state conditions. When these conditions hold – prove that $\frac{\partial z}{\partial x} = -\frac{D_x F}{D_z F}$ and give a formula for $\frac{\partial z}{\partial y}$ under the corresponding condition.

2. For which values of a is $\{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 - z^2 = a\}$ locally a graph? Draw the graph for $a = 1$ and $a = -1$. What "goes wrong" for $a = 0$?

Optimization: Lagrange[2] Multiplier and Kuhn-Tucker methods

Fri, 27.11., 12.30 – 16

Lecture

Aims and Content

Optimization of some function - the cost, the energy, ... with respect to boundary conditions is used in economics, in engineering etc. As a consequence of the implicit function theorem, we get the Lagrange method for optimization with side conditions. It provides necessary conditions for a point to be an extreme point for the restriction of a function f to a subset, if this subset S is defined in terms of some equations $g_i(x_1, x_2, \dots, x_n) = 0$.

In case the subset S is defined by equations *and inequalities*, the Kuhn-Tucker method yields necessary conditions for extreme points of f with respect to S.

References

- **W**, Section 11.7.
- `http://are.berkeley.edu/courses/ARE211/currentYear/lecture_notes/mathNPP1-09-draft.pdf`

Exercises

1. **W**: 11.7.3 a) & c).

2. Let **A** denote a *symmetric* $n \times n$ matrix. Prove that
$$\mu := \max\{\mathbf{x} \cdot \mathbf{A}x| \ \| \mathbf{x} \| \ = 1\}$$
is an eigenvalue of **A**, i.e., there is a vector $\mathbf{y} \in \mathbf{R}^n$ such that $\mathbf{A}y = \mu\mathbf{y}$. In particular, every symmetric matrix has real eigenvalues.
(Hint: The Lagrange multiplier condition on the maximum of the function $f : \mathbf{R}^n \to \mathbf{R}^n$, $f(\mathbf{x}) = \mathbf{x} \cdot \mathbf{A}\mathbf{x}$ under the side condition $\mathbf{x} \cdot \mathbf{x} = 1$. The symmetry of A yields a simple expression for Df.)

3. Find the maximum of $(x_1 x_2 \cdots x_n)^2$ under the restriction
$$x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

[2] `http://www-groups.dcs.st-and.ac.uk/~history/Mathematicians/Lagrange.html`

Course Evaluation

The present course

Please, give us your comments and proposals (for another try) about

- the form of the course (lectures, exercises etc.)
- the literature
- the accessibility and the workload
- the relevance
- etc

Demands for other math courses at the Ph.D.-level

All sorts of math courses that you have always dreamt about but never dared to ask for...

Bye-bye

Finally, we would like to thank you for your kind and active participation.

Best regards,

Lisbeth and Morten.