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Solution to exercize Kreyszig 10.2.11

a) We verify that $f = \arctan(y/x)$ is indeed a potential function by:

$$\frac{\partial f}{\partial x} = \frac{1}{1+(y/x)^2}\,\frac{-y}{x^2} = \frac{-y}{x^2+y^2} = F_1$$
$$\frac{\partial f}{\partial y} = \frac{1}{1+(y/x)^2}\,\frac{1}{x} = \frac{x}{x^2+y^2} = F_2$$

b)We notice, however, that this potential has some discontinuities as:

(i) approaching the y-axis ($x \to 0$) from the 1st and 2nd quadrant gives, respectively:

$$\lim_{x\downarrow 0} f = \frac{\pi}{2} \ : \ x > 0 \wedge y > 0$$
$$\lim_{x\uparrow 0} f = -\frac{\pi}{2} \ : \ x < 0 \wedge y > 0$$

We could fix this discontinuity by adding π to f in the 2nd quadrant.

(ii) approaching the y-axis from the 3rd and 4th quadrant gives, respectively:

$$\lim_{x\downarrow 0} f = -\frac{\pi}{2} \ : \ x > 0 \wedge y < 0$$
$$\lim_{x\uparrow 0} f = \frac{\pi}{2} \ : \ x < 0 \wedge y < 0$$

We could fix this discontinuity by adding the constant $-\pi$ to f in the 4th quadrant.

So, then:

$$f = \begin{cases} \arctan(y/x) & : \ x > 0 \\ \arctan(y/x) + \pi & : \ x < 0 \ : \ y > 0 \\ \arctan(y/x) - \pi & : \ x < 0 \ : \ y < 0 \end{cases}$$

And we are left with a discontinuity (a jump of 2π) on the negative x-axis (going from the 4th to 3rd quadrant). Please note however, that the discontinuity arising in this case at the negative x-axis is here by construction! By adding a proper constant to f in certain domains, we can construct to have the discontinuity on an abitrary half-line starting from the origin. Think about it!

This 2π jump somewhere in the domain (on a half-line starting from the origin) is exactly why the integration over a full circle as in example 4 in section 10.2 gives 2π as an answer.

Further, it implies that examples of domains were we have path independence are e.g.:

i) individual quadrants

ii) the domain consisting of 2 adjacent quadrants.