Author: J.O. Hooghoudt ; email:janotto@math.aau.dk ; date=05/11/13

Solution to exercize Kreyszig 10.4.5

$F = [x^2 + y^2, x^2 - y^2]$; $R : 1 < y < 2 - x^2$. Please draw the figure, so it is clear which area is defined by R and to understand over what you are integrating! Further it helps to find the correct parameterization of the contour C surrounding the area R and to find the integration boundaries of t!!

$$\int\int_R \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) = \int_{-1}^{1}\left(\int_{1}^{2-x^2} (2x - 2y)\mathrm{d}y\right)\mathrm{d}x \quad (1)$$

$$= \int_{-1}^{1} \left[(2xy - y^2)\right]_{y=1}^{y=2-x^2} \mathrm{d}x \quad (2)$$

$$= \int_{-1}^{1} \left(\left(2x(2 - x^2) - (2 - x^2)^2\right) - \left(2x - (1)^2\right)\right)\mathrm{d}x \quad (3)$$

$$= \int_{-1}^{1} \left(4x - 2x^3 - x^4 + 4x^2 - 4 - 2x + 1\right)\mathrm{d}x \quad (4)$$

$$= \int_{-1}^{1} \left(-x^4 - 2x^3 + 4x^2 + 2x - 3\right)\mathrm{d}x \quad (5)$$

$$= \left[-\frac{1}{5}x^5 - \frac{1}{2}x^4 + \frac{4}{3}x^3 + x^2 - 3x\right]_{-1}^{1} \quad (6)$$

$$= -\frac{2}{5} + \frac{8}{3} - 6 = -\frac{56}{15} \quad (7)$$

To calculate the line integral $\oint_C F_1\mathrm{d}x + F_2\mathrm{d}y$:

i) Parameterize the arc as.

$$x = t\,;\, y = 2 - t^2 \;:\; \text{ and so } \frac{\mathrm{d}x}{\mathrm{d}t} = 1 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = -2t \text{ and}$$
$$F_1 = (t^2) + (2 - t^2)^2 = t^4 - 3t^2 + 4$$
$$F_2 = (t^2) - (2 - t^2)^2 = -t^4 + 5t^2 - 4$$

with $t = [1, -1]$ as we have to integrate counter clockwise.

ii) parameterize the line as

$$x = t\,;\, y = 1 \;:\; \text{ and so } \frac{\mathrm{d}x}{\mathrm{d}t} = 1 \text{ and } \frac{\mathrm{d}y}{\mathrm{d}t} = 0 \text{ and}$$
$$F_1 = (t^2) + (1)^2 = t^4 + 1$$
$$F_2 = (t^2) - (1)^2 = -t^4 - 1$$

with $t = [-1, 1]$ (as again we have to integrate counter clockwise). Then:

$$\begin{aligned}
\oint_C F_1 \mathrm{d}x + F_2 \mathrm{d}y &= \oint_C \big(F_1(t)x' + F_2 y'\big)\,\mathrm{d}t \\
&= \int_1^{-1} (F_1(t)1 + F_2(-2t))\,\mathrm{d}t + \int_{-1}^{1} (F_1(t)1 + F_2(t))\,\mathrm{d}t \\
&= \int_1^{-1} \big((t^4 - 3t^2 + 4)\cdot 1 + (-t^4 + 5t^2 - 4)(-2t)\big)\,\mathrm{d}t + \int_{-1}^{1} \big((t^2+1)\cdot 1 + (t^2-1)\cdot(0)\big)\,\mathrm{d}t \\
&= \int_1^{-1} \big((2t^5 + t^4 - 10t^3 + -3t^2 + 8t + 4)\big)\,\mathrm{d}t + \int_{-1}^{1} (t^2+1)\mathrm{d}t \\
&= \left[(\frac{1}{3}t^6 + \frac{1}{5}t^5 - \frac{10}{4}t^4 + -t^3 + 4t^2 + 4t)\right]_1^{-1} + [\frac{1}{3}t^3 + t]_{-1}^{1}\mathrm{d}t \\
&= \left(-\frac{2}{5} + 2 - 8\right) + \left(\frac{2}{3} + 2\right) \\
&= -\frac{56}{5}
\end{aligned}$$

So Green's theorem is verified: the surface integral and the line integral give the same solution.