Resumé: 1° $X'(t) = A_{n,n}\cdot X(t)$ løses af $\underline{X(t) = e^{tA}X_0}$, $X_0 \in \mathbb{R}^n$

Her er $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!}A^k$; og $\Phi(t) = e^{tA}$ er fundamentalmatrix

dvs $\Phi'_{n,n}(t) = A\cdot \Phi_{n,n}(t)$, $\Phi(t)$ inverterbar for alle t (det $\Phi(t) \neq 0$).

NB! $e^{tA} = \mathcal{Y}(t)\cdot\mathcal{Y}(0)^{-1}$, for en vilkårlig fundamentalmatrix

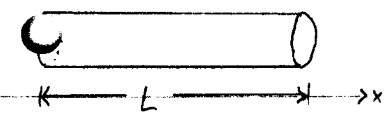
$e^{tA} = e^{t\lambda}\cdot e^{tN}$, når $A = \lambda I + N$, hvor N er nilpotent

2°. $\left.\begin{array}{l} X'(t) = AX(t) + f(t) \\ X(0) = X_0 \end{array}\right\}$ løses af $X(t) = e^{tA}X_0 + \int_0^t e^{(t-s)A}f(s)\,ds$.

NB! Det væsentlige er at løse den homogene ligning først! (jvf. e^{tA}).

Varmeledning:

$u(x,t)$ = temperaturen



Varmeflux

$\varphi(t,x) = K\frac{\partial u}{\partial x}(x,t)$

$\left[\frac{J}{s\cdot m^2}\right]$

$$(V)\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} , & \text{for } 0<x<L \text{ og } 0<t \\ u(0,t) = 0 , & \text{for } t>0 \\ u(L,t) = 0 , & \text{for } t>0 \\ u(x,0) = f(x) , & \text{for } 0<x<L \end{cases}$$

Bem: 1° Problemets data $f(x)$ er en given funktion (ikke bare en konstant)

2° Randbetingelserne $u(0,t) = 0 = u(L,t)$ (konstant temp.) definerer den homogene Dirichlet randbetingelse.

Ex 8.5.1: $u_t = u_{xx}$, $u(0,t) = 0 = u(\pi,t)$, $u(x,0) = 80\sin^3 x = 60\sin x - 20\sin 3x$.

$u_1 = e^{-t}\sin x$, $u_2 = e^{-4t}\sin 2x$, $u_3 = e^{-9t}\sin 3x$ er løsningen (Prøv efter)

Enhver lin. komb. ($c_1, c_2, c_3 \in$ R eller C) er løsning: $u = c_1u_1 + c_2u_2 + c_3u_3$

lineær) Thi $\mathbb{L} = \frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}$ opfylder $\mathbb{L}(c_1u_1 + c_2u_2) = \frac{\partial}{\partial t}(c_1u_1 + c_2u_2) - \frac{\partial^2}{\partial x^2}(c_1u_1 + c_2u_2) = c_1\frac{\partial u_1}{\partial t} + c_2\frac{\partial u_2}{\partial t} + \dots = c_1\mathbb{L}u_1 + c_2\mathbb{L}u_2$

$u = c_1u_1 + c_2u_2 + c_3u_3$ opfylder også randbetingelsen (da u_1, u_2, u_3 gør det)

Mangler at $u(x,0) = 80\sin^3 x$, dvs. at

$$c_1\sin x + c_2\sin(2x) + c_3\sin(3x) = 60\sin x - 20\sin(3x) : \quad \underline{c_1 = 60,\ c_2 = 0}\quad \underline{c_3 = -20.}$$

Altså: $\underline{u(x,t) = 60e^{-t}\sin x - 20e^{-9t}\sin(3x)}$ er løsning(en).

Metode (1) Antag $u_1(x,t), u_2(x,t), \dots$ alle løser (V) og at $u(x,t) = \sum_{n=1}^{\infty} c_n\cdot u_n(x,t)$ er defineret, kontinuert og differentiabel "ledvist" (1 gang mht. t, 2 gange i x)

(2) Antag $f(x) = \sum_{n=0}^{\infty} c_n u_n(x,0)$

(3) Antag $u(x,t) = \sum_{n=0}^{\infty} c_n u_n(x,t)$ forlænger kontinuert til $x=0$, $x=L$, $t=0$.

Da løses (V) af den forlængede funktion $u(x,t)$.

ledr. $u_1, u_2, \dots$ Antag en løsning $u(x,t)$ har separerede variable

$$u(x,t) = X(x)\cdot T(t).$$

Så er (hvis $X \neq 0, T \neq 0$ — — — mål i Kelvin!)

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \Leftrightarrow X(x)T'(t) = kX''(x)T(t) \Leftrightarrow \frac{T'(t)}{kT(t)} = \frac{X''(x)}{X(x)}$$

Da x, t uafhængige, findes i så fald $\lambda \in \mathbb{R}$ ($u(x,t), X, T$ reelle)

$$\frac{T'(t)}{kT(t)} = -\lambda = \frac{X''(x)}{X(x)} \quad \text{for alle } x, t.$$

Dermed <u>TO EGENVÆRDIPROBLEMER</u> for SDL:

$$\begin{cases} X'' + \lambda X = 0 \\ X(0) = 0 = X(L) \end{cases} ; \qquad \begin{cases} T' + \lambda k T = 0 \\ \end{cases}$$

$\lambda = \alpha^2 > 0$: $X(x) = A\cos(\alpha x) + B(\sin \alpha x)$; $X(0) = 0 \Leftrightarrow A = 0$ $X(L) = 0 \Leftrightarrow \alpha L = n\cdot\pi \Leftrightarrow \alpha = n\frac{\pi}{L}$

[$\lambda = -\alpha^2 \leq 0$: $X(x) = A\cosh(\alpha x) + B\sinh(\alpha x)$: $X(0) = 0 \Leftrightarrow A = 0$ $X(L) = 0 \Leftrightarrow B = 0$.]

Dvs: $X(x) = B\cdot \sin(n\frac{\pi}{L}\cdot x)$, $n = 1, 2, 3, \ldots$, $B \in \mathbb{R}$.

$$T' + n^2\frac{\pi^2}{L^2}kT = 0 \quad \Rightarrow \quad T(t) = C\exp(-n^2\frac{\pi^2}{L^2}kt)$$

Altså $u_n(x,t) = \exp(-n^2\frac{\pi^2}{L^2}kt)\sin(n\frac{\pi}{L}x)$, $n = 1, 2, 3\ldots$

vedr. $u(x,0) = f(x)$: $f(x) = \sum_{n=1}^{\infty} c_n \sin(n\frac{\pi}{L}x)$,

med Fourierkoefficient: $c_n = \frac{2}{L}\int_0^L f(x)\sin(n\frac{\pi}{L}x)dx$

<u>Sætning</u>: (V) har løsningen $\underline{u(x,t) = \sum_{n=1}^{\infty} c_n \exp(-n^2\frac{\pi^2}{L^2}kt)\sin(n\frac{\pi}{L}x)}$.

~ ⋅ ~

Isolerede endepunkter: $flux = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$

<u>Neumann-problem</u>:

$$(V)\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} & \text{for } 0 < x < L,\ t > 0 \\ \frac{\partial u}{\partial x}(0,t) = 0 = \frac{\partial u}{\partial x}(L,t) & \text{for } t > 0 \\ u(x,0) = f(x) \end{cases}$$

Her:

$$\begin{cases} X'' + \lambda X = 0 \\ X'(0) = 0 = X'(L): \end{cases}$$

$\lambda = 0$: $X(x) = Ax + B$ $X'(0) = 0 \Rightarrow A = 0$

f.ex $X(x) \equiv 1$. $T(t) \equiv 1$ ($\lambda = 0$)

$\lambda = \alpha^2 > 0$: $X(x) = A\cos(\alpha x) + B\sin(\alpha x)$

$X' = -A\sin(\alpha x) + B\cos(\alpha x)$: $X'(0) = 0 \Rightarrow B = 0$

$X'(L) = 0 \Rightarrow L\alpha = n\cdot\pi \Rightarrow \alpha = n\cdot\frac{\pi}{L}$

<u>Sætning</u>: Neumann problemet (V) har løsningen

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \exp(-n^2\frac{\pi^2}{L^2}kt)\cos(n\frac{\pi}{L}x)$$

idet a_n er Fourierkoefficienterne for dataene $f(x)$:

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\frac{\pi}{L}x)$, $a_n = \frac{2}{L}\int_0^L f(x)\cos(n\frac{\pi}{L}x)dx$, $n = 0, 1, 2, \ldots$