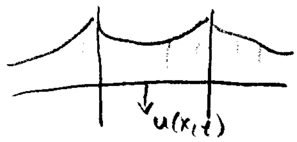
4. S×PDL. 6/3/2006



1-dim bølgeligning:

a = bølgehastigheden.

Cauchy -betingelsen

$$(B)\begin{cases} \dfrac{\partial^2 u}{\partial t^2} = a^2 \dfrac{\partial^2 u}{\partial x^2} & \text{for } 0<x<L,\ 0<t \\ u(0,t) = 0 = u(L,t) & \text{for } t>0 \\ u(x,0) = f(x) & \\ \dfrac{\partial u}{\partial t}(x,0) = g(x) & \text{for } 0<x<L. \end{cases}$$

3-dim. bølgeligning, udsving $u = u(x,y,z,t)$:

(lyd, jordskælv, elektromagnetisme...) $$\frac{\partial^2 u}{\partial t^2} = a^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = a^2 \Delta u$$

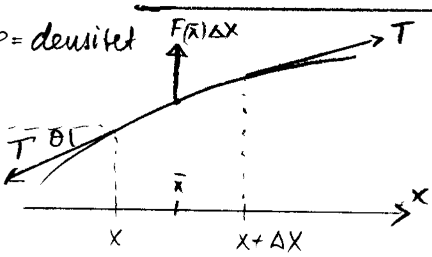
Vedr. a = bølgehastigheden:

d'Alemberts løsning: $F(x)$ 2 gange differentiabel $\Rightarrow \begin{cases} u(x,t) = F(x+at) + F(x-at) \\ \text{løser } u_{tt} = a^2 u_{xx} \end{cases}$

opg. .6.13

$F(x \pm at)$ = bølge mod højre, hastighed $\mp a$.

ρ = densitet



Newton II: $$m\frac{\partial^2 u}{\partial t^2}(\bar{x},t) \approx F(\bar{x})\Delta x + T\sin(\theta+\Delta\theta) - T\sin\theta$$

dvs $$\Delta x \cdot \rho \cdot \frac{\partial^2 u}{\partial t^2}(\bar{x},t) \approx F(\bar{x})\Delta x + T\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right)$$

da $\frac{\sin\theta}{\tan\theta} = \cos\theta = 1 - \frac{1}{2}\theta^2 + \dots$, så $\sin\theta \approx \tan\theta$ for $|\theta| \approx 0$.

Derfor: $$\frac{\partial^2 u}{\partial t^2}(\bar{x},t) \approx \frac{1}{\rho}F(\bar{x}) + T\cdot\frac{1}{\rho}\cdot\frac{1}{\Delta x}\left(\frac{\partial u}{\partial x}(x+\Delta x,t) - \frac{\partial u}{\partial x}(x,t)\right)$$

for $\Delta x \to 0$: $$\boxed{\frac{\partial^2 u}{\partial t^2}(x,t) = \frac{1}{\rho}F(x) + \frac{1}{\rho}T\cdot\frac{\partial^2 u}{\partial x^2}(x,t).}$$

NB! $\boxed{a^2 = \frac{T}{\rho} \quad \left[\frac{m}{s}\right]}$, $\frac{F}{\rho}$ = kraft pr. vægt.

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fart-data $g(x) \equiv 0$: $$u(x,t) = \sum_{n=1}^{\infty} A_n \cos\left(n\cdot\frac{\pi a}{L}\cdot t\right)\sin\left(n\cdot\frac{\pi}{L}\cdot x\right).$$

hvor $$f(x) = \sum_{n=1}^{\infty} A_n \sin\left(n\cdot\frac{\pi}{L}\cdot x\right) \quad \text{(sinusrække for } f\text{)}$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(n\cdot\frac{\pi}{L}\cdot x\right)dx.$$

Ex 1:

$u_{tt} = 4u_{xx}$, for $0<x<\pi$, $t>0$

$u(0,t) = u(\pi,t) = 0$ for $t>0$

$u(x,0) = \frac{1}{10}\sin^3 x$

$u_t(x,0) \equiv 0$.

$$\sin^3(x) = \left(\frac{1}{2i}(e^{ix} - e^{-ix})\right)^3 = \frac{-1}{8i}\left(e^{i3x} - 3e^{ix} + 3e^{-ix} - e^{-i3x}\right) = -\frac{1}{4}\sin(3x) + \frac{3}{4}\sin(x).$$

Deraf: $\left.\begin{matrix} A_1 = 3/40 \\ A_3 = -1/40 \end{matrix}\right\}$ så

$$\underline{\underline{u(x,t) = \frac{3}{40}\cos(2t)\sin x - \frac{1}{40}\cos(6t)\sin(3x).}}$$

Obs: $\mathbb{L}(u) = \frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2}$ *lineær* differential operator.
For $F(x) \equiv 0$ derfor nok at addere løsningerne til

Problem A: $u_{tt} = a^2 u_{xx}$, $u(0,t) = 0 = u(L,t)$, $u(x,0) = f(x)$, $u_t(x,0) = 0$.

Problem B: ——— // ——— $u(x,0) = 0$, $u_t(x,0) = g(x)$

Vedr A: Antag $u(x,t) = X(x) \cdot T(t) \not\equiv 0$ løser bølgeligningen + randbetingelser.

$$u_{tt} = a^2 u_{xx} \Rightarrow X \cdot T'' = a^2 X'' \cdot T$$

$$\Rightarrow \frac{T''}{a^2 T} = \frac{X''}{X} = -\lambda \quad (\lambda \text{ et fast tal}).$$

Deraf:

$$X'' + \lambda X = 0 \qquad\qquad T'' + \lambda a^2 T = 0$$
$$X(0) = 0$$
$$X(L) = 0$$

da $0 = u(0,t) = X(0)\,T(t)$
$0 = u(L,t) = X(L)\,T(t)$ $\quad T(t) \not\equiv 0$

$0 = u_t(x,0) = X(x)\,T'(0)$

$\Rightarrow T'(0) = 0$ (da $X \not\equiv 0$).

Sidste gang: $\lambda_n = \frac{n^2\pi^2}{L^2}$

$X_n(x) = \sin(n \cdot \frac{\pi}{L} \cdot x)$, $n = 1, 2, \ldots$

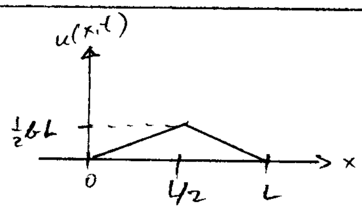
$T(t) = A_n \cos(n \frac{\pi a}{L} t) + B_n \sin(n \frac{\pi a}{L} t)$

$T'(0) \Rightarrow B_n = 0 \;\; \forall n$.

Superposition (da $\mathbb{L}$ er lineær)

$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n \cdot \tfrac{\pi a}{L} t) \sin(n \cdot \tfrac{\pi}{L} \cdot x), \quad A_n \text{ Fourier koeff. for } u(x,0) = f(x).$$

Ex 2: udslået streng



$$u(x,t) = \sum_{n=1}^{\infty} A_n \cos(n \cdot \tfrac{\pi a}{L} t) \sin(n \cdot \tfrac{\pi}{L} \cdot x)$$

frekvens $= \nu_n = \frac{n \cdot \pi \cdot a/L}{2\pi} = \frac{na}{2L} = \underline{\frac{n}{2L}\sqrt{\frac{T}{\rho}}}$

ν_n overtoner, egenfrekvens $= \nu_1 = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$.

Advarsel ledvis diff. normalt umuligt !!

Ex 2 giver $u_{xx} = \sum_{1}^{\infty} -\frac{4b}{L} \sin(\frac{n\pi}{2}) \cos(n \frac{\pi a}{L} t) \sin(n \cdot \frac{\pi}{L} x)$: $\frac{1}{n^2}$ væk !!

Men $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

giver $u(x,t) = \frac{1}{2} \sum_{1}^{\infty} A_n \sin(n \frac{\pi}{L}(x+at)) + \frac{1}{2} \sum_{n=1}^{\infty} A_n \sin(n \frac{\pi}{L}(x-at)) = \frac{1}{2}F(x+at) + \frac{1}{2}F(x-at)$

— som er løsning, endda d'Alemberts løsning

Stedsdata $\equiv 0$

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(n \cdot \tfrac{\pi a}{L} \cdot t) \sin(n \cdot \tfrac{\pi}{L} \cdot x)$$

hvor

$$B_n = \frac{2}{n\pi a} \int_0^L g(x) \sin(n \cdot \tfrac{\pi}{L} \cdot x)\, dx.$$