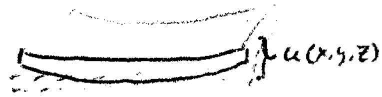
5.3 & PDL 10/3/2006

Laplaceligen: $\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$, skrives også $\Delta u = 0$.

Ex 1: Stationær temperaturfordeling $0 = \frac{\partial u}{\partial t} = k\Delta u$ (motorblok?)

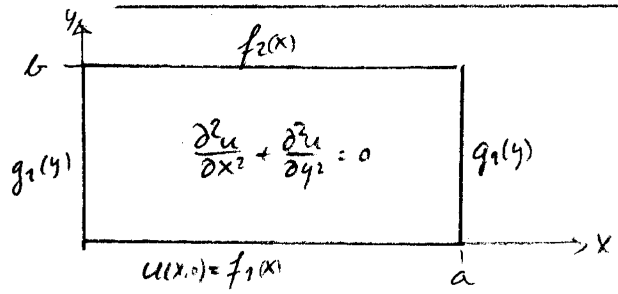
Ex 2: Elektrostatisk potentiel φ: $\Delta\varphi = 0$ $0 < q$ $V = 1000$ volt $F = \nabla\varphi$.

Ex 3: Elasticitet: (små deformationer)



$\Delta^2 u = 0$ dvs $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 0$

Stationær temp:



$\Delta u = 0$

$u(x,0) = f_1(x)$, $u(x,b) = f_2(x)$ for $0 < x < a$

$u(0,y) = g_1(y)$, $u(a,y) = g_2(y)$ for $0 < y < b$

Løsning: $u = u_1 + u_2 + v_1 + v_2$, hvor u_1 løser pb. med kun $f_1 \neq 0$ etc. mens v_1 løser pb. med kun $g_1 \neq 0$ etc.
(da Δ er lineær)

Ex 1: $f_1(x)$ given, mens g_1, g_2, f_2 alle er $\equiv 0$.

Separation af de variable: Hvis $u(x,y) = X(x)Y(y) \neq 0$ er løsning ses at

$$\Delta u = 0 \iff \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \iff X''Y + XY'' = 0$$

$$\iff \exists -\lambda \in \mathbb{C}: \quad \frac{X''}{X} = -\lambda = -\frac{Y''}{Y}$$

Derfor

$X'' + \lambda X = 0$	$Y'' - \lambda Y = 0$
$X(0) = 0$	$Y(b) = 0$
$X(a) = 0$	

da $u(0,y) \equiv 0 \equiv u(a,y)$ oversættes til $X(0)Y(y) \equiv 0 \equiv X(a)Y(y)$

Som sædv. $X(x) = \sin(n\cdot\frac{\pi}{a}\cdot x)$ og $\lambda_n = \frac{n^2\pi^2}{a^2}$, for $n = 1, 2, \ldots$

Men $Y''(y) - \lambda_n Y(y) = 0$, $Y(b) = 0$

har karakterlign.

$$r^2 - \lambda_n = 0$$

Dvs $r_n = \pm\sqrt{\lambda_n} = \pm\frac{n\pi}{a}$, $n = 1, 2, \ldots$

Dermed

$$Y_n(y) = A_n \cosh(n\tfrac{\pi}{a}y) + B_n \sinh(n\cdot\tfrac{\pi}{a}\cdot y).$$

Eller $\quad Y_n = a_n e^{r_n y} + b_n e^{-r_n y}$

Men $\quad 0 = Y_n(b) = a_n e^{r_n b} + b_n e^{-r_n b}$

$$\Rightarrow \quad b_n = -a_n e^{r_n b} \cdot e^{r_n b} = -a_n e^{2 r_n b}$$

Så

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$$Y_n = a_n (e^{r_n y} - e^{2r_n b} e^{-r_n y})$$
$$= a_n e^{r_n b}\left(e^{r_n (y-b)} - e^{-r_n(y-b)}\right) = 2a_n e^{r_n b} \sinh(r_n(y-b))$$
$$= \text{konstant} \cdot \sinh\left(n \cdot \tfrac{\pi}{a}(b-y)\right)$$

Løsningen

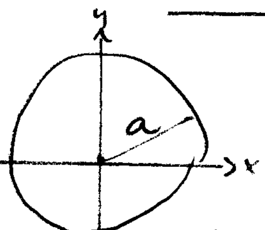
$$u_1(x,y) = \sum_{n=1}^{\infty} X_n(x)\, Y_n(y) = \sum_{n=1}^{\infty} c_n \sin\left(n \cdot \tfrac{\pi}{a} x\right) \sinh\left(n \cdot \tfrac{\pi}{a}(b-y)\right).$$

Ønsker $\quad u(x,0) = \sum_{n=1}^{\infty} c_n \sin\left(n\cdot\frac{\pi}{a}\cdot x\right)\cdot \sinh\left(n\cdot \frac{\pi}{a} b\right) = f_1(x)$

jvf H(14) side 603

Så $\quad c_n \sinh\left(n\cdot\frac{\pi}{a}\cdot b\right) = \frac{2}{a}\int_0^a f(x) \sin\left(n\cdot\frac{\pi}{a}\cdot x\right) dx$.

NB! u_2, v_1 og v_2 fås af opgave 8.7.1-3.
— denæst anvendes sætningen!



$u(x,y) = u(r\cos\theta, r\sin\theta) \quad$ = funktion af r, θ

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

Dirichlet randbetingelse: $\quad u(a,\theta) = f(\theta) \qquad$ ($r = a$ på randen)

f 2π-periodisk $\longrightarrow$ $\underline{u(r,\theta) = u(r,\theta+2\pi)}$ (konvention)

Separation r, θ

Antag $\quad u(r,\theta) = R(r)\,\Theta(\theta) \neq 0$

$$\Delta u = 0 \iff R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \qquad \frac{R\Theta}{r^2}$$
$$\iff \frac{R''}{R}r^2 + r\frac{R'}{R} + \frac{\Theta''}{\Theta} = 0$$
$$\iff \exists \lambda \in \mathbb{C}: \quad \frac{1}{R}(r^2 R'' + rR') = \lambda = -\frac{\Theta''}{\Theta}$$

$r^2 R'' + rR' - \lambda R = 0$

$\Theta'' + \lambda\Theta = 0 \qquad (z^2 + \lambda = 0)$

$\Theta(\theta) = A\cos(\alpha\theta) + B\sin(\alpha\theta) \quad$ for $\lambda = \alpha^2 > 0$

$\Theta(\theta) = A + B\theta \quad$ for $\lambda = 0$

$\Theta(\theta) = Ae^{\alpha\theta} + Be^{-\alpha\theta} \quad$ for $\lambda = -\alpha^2 < 0$

Θ 2π-periodisk: $\lambda < 0$ udelukket

$\lambda = 0$: $B = 0$ så $\Theta = A$ (ellers strengt monoton)

$\lambda > 0$: $\lambda = \alpha^2 = n^2$ for $n \in \{1, 2, 3, \ldots\}$

giver periodisk funktion

(hvis $\alpha \in \mathbb{R} \setminus \mathbb{Z}$ er $\Theta(\theta)$ ej 2π-periodisk)

$$\Theta(\theta) = A_n \cos(n\theta) + B_n \sin(n\theta)$$

For $\lambda = 0$ løses $r^2 R'' + r R' = 0$

af $R_0(r) = C_0 + D_0 \ln r$

Kontinuitet i $r = 0$ $\therefore$ $D_0 = 0$ $R_0 = C_0$ (= konstant)

For $\lambda = n^2$ haves $r^2 R_n'' + r R_n' - n^2 R_n = 0$

Indsættes $R_n = r^k$: $k(k-1) r^2 r^{k-2} + r k r^{k-1} - n^2 r^k = 0$

dvs $(k(k-1) + k - n^2) r^k = 0$

Kræver $n^2 = k(k-1) + k = k^2$

dvs $k = \pm n$.

Altså

$R_n(r) = C_n r^n + D_n r^{-n}$ (vides at være to lin. uafh. løsninger $\to r^n$ og r^{-n})

Kontinuitet i $r = 0$, $D_n = 0$

Endelig $R_n(r) = C_n r^n$

Løsning:

$$u(r, \theta) = \sum_{n=1}^{\infty} R_n(r)\Theta(\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) r^n$$

Randbetingelsen

$$u(a, \theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a^n a_n \cos(n\theta) + a^n b_n \sin(n\theta)) = f(\theta)$$

Fourierrækken for $f(\theta)$:

$$a^n a_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \cos(n\theta)\, d\theta, \quad n = 0, 1, 2, \ldots$$

$$a^n b_n = \frac{1}{\pi} \int_0^{2\pi} f(\theta) \sin(n\theta)\, d\theta.$$