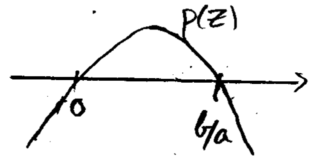
Mat 1, PE: Lin. & Dyn.

4/9/2007

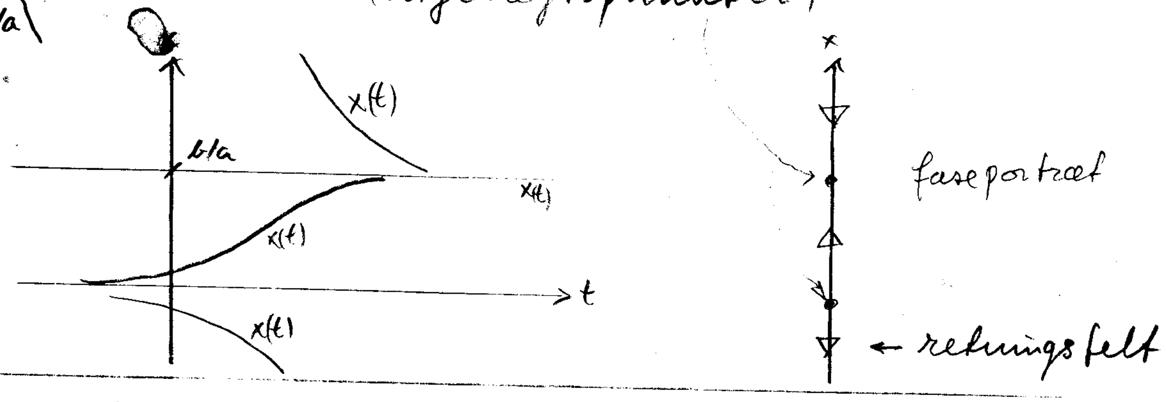
Oversigt over kursets/projektets hovedideer.

<u>Eks</u>: logistisk vækst: $x'(t) = k\,x(t)(N - x(t)) =: a\,x(t) - b\,x(t)^2$, $a, b > 0$.
([AB] 5.8)



rødder i $p(z) = az - bz^2$
giver <u>stationære</u> løsninger
(ligevægtspunkter)

dvs



(3) Slutmålet, <u>1. eks.</u>

$$\frac{dX}{dt} = aX + bX^2 + cXY$$
$$\frac{dY}{dt} = dY + eY^2 + fXY$$

Her:

$$\begin{cases} 0 = az - bz^2 + czw & = z(a + bz + cw) \\ 0 = dw + ew^2 + fzw & = w(d + ew + fz) \end{cases}$$

$\Updownarrow$

$(z = 0 \ \vee \ a + bz + cw = 0)$
$\wedge \, (w = 0 \ \vee \ d + ew + fz = 0)$ (nk. opr.!)

$\Updownarrow$

$(z, w) = (0, 0) \ \vee \ d + ew = 0 \ \vee \ a + bz = 0 \ \vee \begin{cases} bz + cw = -a \\ fz + ew = -d \end{cases}$

<u>Alment</u>:

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}(t)), \qquad \vec{x}(t) \in \mathbb{R}^n$$

dvs

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix} = \begin{bmatrix} f_1(x_1(t), \dots, x_n(t)) \\ \vdots \\ f_n(x_1(t), \dots, x_n(t)) \end{bmatrix}.$$

Ligningen $\vec{f}(\vec{z}) = \vec{0}$ løses mht. $\vec{z} \in \mathbb{R}^n$
(jvf. foregående eksempler)

Altså: <u>n ikke-lineære ligninger med n ubekendte(!)</u>

Jacobi-matricen: $\frac{\partial \vec{f}}{\partial \vec{x}}(\vec{z}) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(\vec{z}) & \cdots & \frac{\partial f_1}{\partial x_n}(\vec{z}) \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(\vec{z}) & \cdots & \frac{\partial f_n}{\partial x_n}(\vec{z}) \end{bmatrix} =: A.$

<u>Sætning</u>: Hvis $\mathrm{Re}\,\lambda < 0$ for enhver egenværdi λ til A,
[AB], kap. 5.4 så er $\vec{z}$ et asymptotisk stabilt ligevægtspunkt
(dvs $\vec{x}(t) \longrightarrow \vec{z}$ for $t \longrightarrow \infty$.)

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② Hjælpeproblem for ③: $\frac{d}{dt}\begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix} = A_{n,n}\begin{bmatrix} y_1(t) \\ \vdots \\ y_n(t) \end{bmatrix}$

$\vec{y}(0) = \vec{v}$

Løsning $\underline{\vec{y}(t) = e^{tA}\cdot\vec{v}}$

Her er $e^{tA} \overset{\text{def}}{=} \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$.

A diagonaliserbar (hovedtilfælde) $\Rightarrow A = PDP^{-1}$

$$\Rightarrow A^k = PD\cancel{P^{-1}}\cancel{P}D\cancel{P^{-1}}\cancel{P}DP^{-1}\ldots PDP^{-1}$$

$$= PD^kP^{-1} = P\begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}P^{-1}$$

$$\Rightarrow e^{tA} = P\cdot\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}\begin{bmatrix} \lambda_1^k & & 0 \\ & \ddots & \\ 0 & & \lambda_n^k \end{bmatrix}\right]\cdot P^{-1}$$

$$= P\cdot\begin{bmatrix} \sum_{k=0}^{\infty}\frac{(t\lambda_1)^k}{k!} & & \\ & \ddots & \\ & & \sum_{k=0}^{\infty}\frac{(t\lambda_n)^k}{k!} \end{bmatrix}P^{-1}$$

$$= P\cdot\begin{pmatrix} e^{t\lambda_1} & & 0 \\ & \ddots & \\ 0 & & e^{t\lambda_n} \end{pmatrix}\cdot P^{-1}.$$

Her gælder jo at

$$e^{t\lambda_j} = e^{Re(t\lambda_j) + i\,Im(t\lambda_j)} = e^{t\,Re\lambda_j}(\cos(t\,Im\lambda_j) + i\sin(t\,Im\lambda_j))$$

$\longrightarrow 0$ for $t \to \infty$, $\underline{\text{dersom } Re\lambda_j < 0}$!!

(jævnfør sætningen)

[også rigtigt når A ej er diagonaliserbar]

① $\underline{\text{1. ordens differensligning}}$: $\vec{x}_{k+1} = A_{n,n}\vec{x}_k$, $\vec{x}_0 = \vec{v} \in \mathbb{R}^n$

Kilde: $\underline{\text{HEJ (basis bog), kap. 7}}$

$\underline{\text{Eks}}$: Kapitalformlen: $K_{n+1} = (1+r)K_n$, r = rentefoden

$$\boxed{K_n = (1+r)^n K_0}$$

$\underline{\text{Eks}}$: $\underline{\text{Løsning af } \vec{f}(\vec{z}) = \vec{0} \text{ i ③}}$:

Taylor til 1. orden: $\vec{f}(\vec{x}) = \vec{f}(\vec{z}) + \frac{\partial\vec{f}}{\partial\vec{x}}(\vec{z})(\vec{x}-\vec{z}) + \vec{R}_1(\vec{x})$

Dvs $\frac{\partial\vec{f}}{\partial\vec{x}}(\vec{z})\cdot(\vec{x}-\vec{z}) \approx \vec{f}(\vec{x}) - \vec{f}(\vec{z})$

så $\vec{x} \approx \vec{z} + \left(\frac{\partial\vec{f}}{\partial\vec{x}}(\vec{z})\right)^{-1}\cdot(\vec{f}(\vec{x}) - \vec{f}(\vec{z}))$

Deraf: $\vec{x}_{k+1} = \vec{x}_k + \frac{\partial\vec{f}}{\partial\vec{x}}(\vec{x}_k)^{-1}\cdot(\vec{f}(\vec{x}_k) - \vec{f}(\vec{x}_{k-1}))$ $\left\{\begin{array}{l}\vec{x}_0, \vec{x}_1 \\ \text{gæt} \\ \text{på} \\ \vec{z}\end{array}\right.$

$\underline{\text{Ikke-lineær}}$ 1. ordens differens lign!!

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Eks: <u>Storplettet ugle</u> (Lay, kap. 5): $\vec{x}_{k+1} = A \cdot \vec{x}_k$

hvor $A = \begin{bmatrix} 0 & 0 & 0{,}33 \\ 0{,}18 & 0 & 0 \\ 0 & 0{,}71 & 0{,}94 \end{bmatrix}$ $\lambda_1 = 0{,}98$, $\lambda_\pm = -\frac{3}{10} \pm i \frac{21}{100}$ $v_1, v_\pm \in \mathbb{C}^3$

$$\vec{x}_k = A^k \vec{x}_0 = A^k (c_1 \vec{v}_1 + c_+ \vec{v}_+ + c_- \vec{v}_-) = c_1 A^k v_1 + c_+ A^k v_+ + c_- A^k v_-$$

$Av = \lambda v$

$$= c_1 \lambda_1^k \vec{v}_1 + c_+ \lambda_+^k \vec{v}_+ + c_- \lambda_-^k \vec{v}_-$$

$$= c_1\, 0{,}98^k \vec{v}_1 + c_+ \left(-\tfrac{3}{10} + i \tfrac{21}{100}\right)^k \vec{v}_+ + c_- \left(-\tfrac{3}{10} - i \tfrac{21}{100}\right)^k \vec{v}_-$$

i $\mathbb{C}^3$: $\|0{,}98^k \vec{v}_1\| = 0{,}98^k \|\vec{v}_1\| \longrightarrow 0$ for $k \to \infty$

NB! „længde" i $\mathbb{C}^3$! $\|(-\tfrac{3}{10} \pm i \tfrac{21}{100})^k \vec{v}_\pm\| = |-\tfrac{3}{10} \pm i \tfrac{21}{100}|^k \cdot \|\vec{v}_\pm\| \longrightarrow 0$ for $k \to \infty$.

Vektor V over skalarlegemet $\mathbb{L} = \mathbb{R}$ el. $\mathbb{L} = \mathbb{C}$

antag $(v, w) \longmapsto v + w$ er en afbildning $V \times V \longrightarrow V$

$(\lambda, w) \longmapsto \lambda w$ — " — $\mathbb{L} \times V \longrightarrow V$

Defn: V kaldes et vektorrum over $\mathbb{L}$, dersom

(i) $u + v = v + u$ for alle $u, v \in V$

(ii) $(u+v)+w = u + (v+w)$ for alle u, v, w

(iii) der eksisterer $0 \in V$ så $v + 0 = v$ for alle $v \in V$

(iv) til hvert $v \in V$ findes $v' \in V$ så $v + v' = 0$.

(v) $1v = v$ for alle $v \in V$

(vi) $\lambda(u+v) = \lambda u + \lambda v$ for alle $\lambda \in \mathbb{L}$, alle $u, v \in V$

(vii) $(\lambda + \mu) v = \lambda v + \mu v$ for alle $\lambda, \mu \in \mathbb{L}$, alle $v \in V$

(viii) $\lambda(\mu v) = (\lambda \mu) v$ for alle $\lambda, \mu \in \mathbb{L}$, alle $v \in V$.

Eks 0: $V = \mathbb{R}^n$. Eks 1: $V = \mathbb{C}^n$ eller $V = \mathbb{L}^n$.

Eks 2: $V = \mathbb{L}^\infty = \{ (x_1, x_2, \dots, x_k, \dots) \mid x_k \in \mathbb{L}$ for alle $k \in \mathbb{N} \}$

med $(x_1, x_2, \dots) + (y_1, y_2, \dots) = (x_1 + y_1, x_2 + y_2, \dots)$

$\lambda (x_1, x_2, \dots) = (\lambda x_1, \lambda x_2, \dots)$

<u>vektorrum</u> over $\mathbb{L}$, da (i)–(viii) gælder for hvert k, $\vec{0} = (0, 0, \dots)$.

Eks 3: $V = \mathcal{P}(\mathbb{L}) = \{ p : \mathbb{L} \to \mathbb{L} \mid p(z) = a_0 + a_1 z + \dots + a_m z^m$ for alle $z \in \mathbb{L} \}$.

<u>vektorrum</u>, da (i)–(viii) gælder for hvert k, $\vec{0} = 0z^1 = 0$.

Eks 4: $V = \mathcal{F}(M, \mathbb{L}) = \{ f : M \to \mathbb{L} \mid f$ er en funktion$\}$ M vilk. mængde.

Eks 5: $V = \mathcal{F}(M, W) = \{ f : M \to W \mid$ — " — $\}$ W givet vektorrum

V er selv et <u>vektorrum</u>, da (i)–(viii) gælder punktvis for hvert $m \in M$, f. eks.

$$(f+g)(m) = f(m) + g(m) \overset{i\,W}{=} g(m) + f(m) = (g+f)(m), \text{ så } f+g = g+f.$$

<u>Sætning</u>: 0 entydigt bestemt, modsatte vektorer (dvs. v') entydigt best.
$0 \cdot v = 0$ for alle v; $\lambda \cdot 0 = 0$ for alle λ. $(-1)\vec{v} = \vec{v}$ for alle v.

<u>Defn</u>: $U \subseteq V$ er <u>underrum</u>, hvis $U \neq \emptyset$ og stabil under + og $\cdot$.

<u>Sætn</u>: U er <u>vektorrum</u>, hvis U er et underrum.

Eks: $\mathcal{P}(\mathbb{L}) \subseteq \mathcal{F}(\mathbb{L}, \mathbb{L})$ er et underrum.