NOTES ON CALCULUS OF VARIATIONS

JON JOHNSEN

October 3, 2006

The basic problem

In Calculus of variations one is given a fixed C^2-function $F(t, x, u)$, for $t \in [t_0, t_1]$ and $x, u \in \mathbb{R}$, and the problem is to maximise (or minimise) the functional

$$J(x) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt \tag{1}$$

that acts on functions $x\colon [t_0, t_1] \to \mathbb{R}$ fulfilling

$$x(t_0) = x^0, \qquad x(t_1) = x^1. \tag{2}$$

Hereby x^0, x^1 are given numbers; and any C^1-function[1] $x(t)$ satisfying these two boundary conditions is said to be *admissible*.

Example 1 (Ramsey 1928)**.** In economics it is a task to determine the amount of capital $K(t)$, say in a country, such that, when $f(K(t))$ denotes the gross domestic product created by having the capital $K(t)$, one has

$$\max \int_0^T U(f(K(t)) - \dot{K}(t)) e^{-\rho t}\, dt, \qquad K(0) = K_0, \quad K(T) = K_T. \tag{3}$$

Here $f(K) - \dot{K}$ is the consumption and U is a *utility* function, fixed so that $U' > 0 > U''$ on $]0, \infty[$. The factor $e^{-\rho t}$, $\rho > 0$, gives a discount of future consumption in order to give priority to consumption in the near future. K_0 and K_T are given initial and terminal values of the available capital. By maximising the integral, the country is envisaged to benefit as much as possible from the change in capital from K_0 to K_T.

The problem above can be seen as an optimisation problem in infinitely many variables (one for each $t \in [t_0, t_1]$). But fortunately the possible minimising and maximising functions $x(t)$ can be found among the solutions to a certain ordinary differential equation. This is the content of the following famous result:

[1] Recall that a function $f(t)$ is said to be a C^k-function on an interval $[t_0, t_1]$ if all derivatives $\frac{d^m f}{dt^m}$ with $0 \le m \le k$ exist and are continous on $[t_0, t_1]$. The analogous definition applies to functions of several variables, such as $F(t, x, u)$.

Theorem 2 (Euler–Lagrange equation)**.** *For an admissible function $x^*(t)$ to maximise or minimise*

$$J(x) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt, \quad \text{such that} \quad x(t_0) = x^0, \quad x(t_1) = x^1 \tag{4}$$

it is necessary that $x^*(t)$ solves the ordinary differential equation

$$\tfrac{d}{dt}(F_3'(t, x(t), \dot{x}(t))) = F_2'(t, x(t), \dot{x}(t)). \tag{5}$$

To reveal the nature of the above differential equation, it is instructive to carry out the t-differentiation using the chain rule (this is allowed at least if the solution $x(t)$ is a C^2-function):

$$F_{33}''(t, x(t), \dot{x}(t))\ddot{x}(t) + F_{32}''(t, x(t), \dot{x}(t))\dot{x}(t) - F_2'(t, x(t), \dot{x}(t)) = 0. \tag{6}$$

This is a homogeneous second order differential equation, which is said to be quasi-linear because the coefficients depend on the solution and its lower order derivatives.

To illustrate the usefulness of Theorem 2, one can take the simple

Example 3. Consider

$$J(x) = \int_0^1 (x(t)^2 + \dot{x}(t)^2)\, dt; \qquad x(0) = 0, \quad x(1) = e^2 - 1.$$

Here $F(x, u) = x^2 + u^2$ so the Euler–Lagrange equation is

$$0 = \tfrac{d}{dt}(2\dot{x}) - 2x = 2(\ddot{x} - x).$$

The complete solution to this is given by the functions $x(t) = Ae^t + Be^{-1}$. Invoking the boundary conditions it is seen that $x(t)$ is admissible precisely for $A = e = -B$, so the only *candidate* for a minimiser or maximiser is

$$x^*(t) = e^{1+t} - e^{1-t}.$$

However, x^* is not a maximising function ($J(x)$ can be seen to take on arbitrarily large values), but it will follow later from sufficient conditions that x^* is a minimiser.

Proof of the Euler–Lagrange equation. The point of departure is to show what is known as the *fundamental lemma of calculus of variations*:

Lemma 4. *Let $f\colon [t_0, t_1] \to \mathbb{R}$ be a continuous function with the property that $\int_{t_0}^{t_1} f(t)\mu(t)\, dt = 0$ for every C^2-function $\mu(t)$ on $[t_0, t_1]$ such that $\mu(t_0) = 0 = \mu(t_1)$. Then $f(t) = 0$ for every t.*

Proof. Suppose $f(s) \neq 0$, say $f(s) > 0$. By continuity there is some interval $I =]a, b[$ on which $f(t) > 0$ and $t_0 < a < b < t_1$. On I one can then define $\mu(t)$ as $\mu(t) = (t-a)^3(b-t^3) > 0$, and let $\mu(t) = 0$ outside I; the zeroes at a and b have so high order that this μ is C^2. Now $0 = \int_{t_0}^{t_1} f(t)\mu(t)\, dt = \int_I f(t)\mu(t)\, dt > 0$, which is a contradiction; hence $f \equiv 0$. □

The lemma above is exploited by forming a so-called *variation* of the given solution,

$$x(t,\alpha) = x^*(t) + \alpha\mu(t). \tag{7}$$

Here $\alpha \in \mathbb{R}$ is just a parameter, while μ is an arbitrary C^2-function on $[t_0, t_1]$ such that $\mu(t_0) = 0 = \mu(t_1)$; clearly $t \mapsto x(t,\alpha)$ is then admissible for every fixed α.

As a convenient notation, let

$$I(\alpha) = J(x(t,\alpha)) = \int_{t_0}^{t_1} F(t, x(t,\alpha), \tfrac{\partial x(t,\alpha)}{\partial t})\, dt. \tag{8}$$

For simplicity the proof continues with the case of a maximum at x^* (the case of a minimum is similar). This means that $I(\alpha) \le I(0)$ for all α, whence

$$I'(0) = 0. \tag{9}$$

This is, of course, under the tacit assumption that $\alpha \mapsto I(\alpha)$ is differentiable; it is furthermore assumed that $I'(\alpha)$ is obtained by differentiating under the integral sign above (this is proved later).

Proceeding from this, one arrives at

$$\begin{aligned} I'(0) &= \int_{t_0}^{t_1} (F_2'(t,x^*,\dot{x}^*)\tfrac{\partial x}{\partial \alpha}(t,0) + F_3'(t,x^*,\dot{x}^*)\tfrac{\partial^2 x}{\partial t \partial \alpha}(t,0))\, dt \\ &= \int_{t_0}^{t_1} (F_2'(t,x^*,\dot{x}^*)\mu(t) + F_3'(t,x^*,\dot{x}^*)\dot{\mu}(t))\, dt. \end{aligned} \tag{10}$$

Since μ vanishes at the end points, an integration by parts gives

$$0 = I'(0) = \int_{t_0}^{t_1} \mu(t)(F_2'(t,x^*,\dot{x}^*) - \tfrac{d}{dt}F_3'(t,x^*,\dot{x}^*))\, dt. \tag{11}$$

Since μ is an arbitrary C^2-function vanishing at the end points, it follows from the above Lemma 4 that the function in the parenthesis is 0 for every t. This means that the Euler–Lagrange equation is satisfied by $x^*(t)$.

Now it only remains to account for the statements in the next lemma.

Lemma 5. *The function $I(\alpha)$ is differentiable at $\alpha = 0$ and $I'(0) = \int_{t_0}^{t_1} \frac{\partial}{\partial\alpha} F(t, x(t,\alpha), \frac{\partial x}{\partial t}(t,\alpha))\big|_{\alpha=0}\, dt$.*

Proof. Clearly

$$I(\alpha) - I(0) = \int_{t_0}^{t_1} (F(t, x(t,\alpha), \tfrac{\partial x(t,\alpha)}{\partial t}) - F(t,x^*,\dot{x}^*))\, dt. \tag{12}$$

Keeping t fixed in the interval, one can calculate the integrand as follows:

$$\begin{aligned}
[F(t,x(t,s\alpha),\tfrac{\partial x}{\partial t}(t,s\alpha))]_{s=0}^{s=1} &= \int_0^1 \tfrac{\partial}{\partial s}F(t,x(t,s\alpha),\tfrac{\partial x}{\partial t}(t,s\alpha))\,ds \\
&= \int_0^1 (F_2'(t,x(t,s\alpha),\tfrac{\partial x}{\partial t}(t,s\alpha))\alpha\mu(t) \\
&\qquad + F_3'(t,x(t,s\alpha),\tfrac{\partial x}{\partial t}(t,s\alpha))\alpha\dot\mu(t))\,ds. \qquad (13)
\end{aligned}$$

Reading $\alpha = -\frac{\partial}{\partial s}(\alpha - \alpha s)$, and introducing the short-hand

$$G(\alpha,s,t) = \mu(t)F_2'(t,x(s\alpha,t),\tfrac{\partial x}{\partial t}(s\alpha,t)) + \dot\mu(t)F_3'(t,x(s\alpha,t),\tfrac{\partial x}{\partial t}(s\alpha,t)), \qquad (14)$$

an integration by parts therefore shows that the above expression equals

$$\int_0^1 \alpha(1-s)\tfrac{\partial}{\partial s}G(\alpha,s,t)\,ds - \big[\alpha(1-s)G(\alpha,s,t)\big]_{s=0}^{s=1}. \qquad (15)$$

Clearly the contribution for $s=1$ vanishes, while for $s=0$ one obtains the functions $x^*(t)$ and $\dot x^*(t)$. So by calculating the s-derivatives, the above formulae lead to the fact that

$$\begin{aligned}
0 \le |\tfrac{1}{\alpha}(I(\alpha)-I(0)) &- \int_{t_0}^{t_1} (\mu F_2'(t,x^*,\dot x^*) + \dot\mu F_3'(t,x^*,\dot x^*))\,dt| \\
\le \int_{t_0}^{t_1}\int_0^1 &|\alpha||1-s|\big|\mu^2 F_{22}''(t,x(s\alpha,t),\tfrac{\partial x}{\partial t}(s\alpha,t)) \\
&+ 2\mu\dot\mu F_{23}''(t,x(s\alpha,t),\tfrac{\partial x}{\partial t}(s\alpha,t)) \\
&+ \dot\mu^2 F_{33}''(t,x(s\alpha,t),\tfrac{\partial x}{\partial t}(s\alpha,t))\big|\,dsdt \\
&\le C\alpha \to 0 \quad \text{for} \quad \alpha \to 0. \qquad (16)
\end{aligned}$$

In fact, it is straightforward to see that the constant C can be taken as

$$C = |t_1 - t_0| \max_{[t_0,t_1]}(|\mu| + |\dot\mu|)^2 \max_B(|F_{22}''|,|F_{23}''|,|F_{33}''|). \qquad (17)$$

Hereby the set B denotes the range of

$$\Phi(\alpha,s,t) = (t, x^*(t) + s\alpha\mu(t), \dot x^*(t) + s\alpha\dot\mu(t)) \qquad (18)$$

which is a continuous map $\Phi\colon [-1,1]\times[0,1]\times[t_0,t_1] \to \mathbb{R}^3$. Since the domain of Φ is a *compact* set, so is necessarily its range B. Consequently the functions F_{22}'', F_{23}'' and F_{33}'' are all bounded on B (they are continuous because F itself was assumed to be C^2), whence $C<\infty$.

Altogether (16) accounts for both the differentiability of $I(\alpha)$ at $\alpha = 0$ and that the derivatives can be calculated under the integral sign. □

Further necessary conditions. Sometimes the following necessary condition is also useful.

Theorem 6 (Legendre)**.** *If F is a C^2-function, it is a necessary condition for the functional $J(x) = \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt$ to have an extreme value at an admissible function $x^*(t)$ that*

$$F''_{33}(t, x^*(t), \dot{x}^*(t)) \leq 0 \quad \textit{in case } x^* \textit{ maximises,} \tag{19}$$

$$F''_{33}(t, x^*(t), \dot{x}^*(t)) \geq 0 \quad \textit{in case } x^* \textit{ minimises.} \tag{20}$$

(These inequalities are required to hold for all $t \in [t_0, t_1]$.)

The Legendre condition is often useful, when one wants to show that a solution candidate is *not* a maximiser (or a minimiser). This is elucidated by the next example.

Example 7. In the above Example 3 where $J(x) = \int_0^1 (x^2 + \dot{x}^2)\, dt$ one finds at once that $F''_{33} = 2 > 0$, and this rules out that the admissible function $x^* = e^{1+t} - e^{1-t}$ can be a maximiser. (It is still open whether x^* is a minimiser; this cannot be concluded from the fact that $F''_{33} > 2$.)

One of the possible complications in practice is that there, for good reasons, are further *constraints* on the admissible functions. This can e.g. leave us with the problems of finding a C^1-function $x \colon [t_0, t_1] \to \mathbb{R}$ such that

$$\max \int_{t_0}^{t_1} F(t, x, \dot{x})\, dt; \qquad x(t_0) = x^0,\ x(t_1) = x^1; \tag{21}$$

$$h(t, x(t), \dot{x}(t)) > 0 \quad \text{for all} \quad t \in [t_0, t_1]. \tag{22}$$

Hereby h is a suitable C^1-function defining the constraint. However, one can show that the Euler–Lagrange and Legendre conditions are necessary also for such problems. The next example shows how such constraints can appear.

Example 8. In the above Example 1 from macro economics, one has the integrand $F(t, K(t), \dot{K}(t)) = U(f(K(t)) - \dot{K}(t))e^{-\rho t}$. But the utility function U is typically only a C^2-function on $]0, \infty[$; for example $U(\cdot) = \sqrt{(\cdot)}$ meets the requirements that $U' > 0 > U''$. So here it is natural to impose the constraint that

$$h(t, K(t), \dot{K}(t)) := f(K(t)) - \dot{K}(t) > 0.$$

This is not just a kind of mathematical obstruction (as $U' \to \infty$ at 0, already a C^1-extension of U to the whole axis $\mathbb{R}$ is impossible), for negative values of the consumption $f(K(t)) - \dot{K}(t)$ may not make sense in the model at all.

Problems with general terminal conditions

A more radical change is met, if one considers the problem of having

$$\max \int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt, \qquad x(0) = x^0, \tag{23}$$

together with one of the following *terminal* conditions:

(i) $x(t_1)$ free (t_1 given);
(ii) $x(t_1) \geq x^1$ (t_1 and x^1 given);
(iii) $x(t_1) = g(t_1)$ (t_1 free, but g a given C^1-function).

Correspondingly the admissible functions are now required to be C^1-functions that fulfill the stated initial and terminal conditions. It is clear that a maximising function x^* still fulfills the Euler–Lagrange equations, for x^* also maximises among the admissible functions that satisfy $x(t_1) = x^*(t_1)$. But one has to add some *transversality conditions*:

Theorem 9. *If $x^*(t)$ is an admissible function solving the above maximisation problem, then x^* solves the Euler–Lagrange equation and the corresponding transversality condition,*

(i) $F_3'(t_1, x^*(t_1), \dot{x}^*(t_1)) = 0$*;*
(ii) $F_3'(t_1, x^*(t_1), \dot{x}^*(t_1)) \leq 0$ *(and $= 0$ holds if $x^*(t_1) > x^1$);*
(iii) $F(t_1, x^*(t_1), \dot{x}^*(t_1)) - (\dot{g}(t_1) - \dot{x}^*(t_1))F_3'(t_1, x^*(t_1), \dot{x}^*(t_1)) = 0.$

In case (ii) *the inequality is reversed if x^* solves the minimisation problem.*

Proof. Set $y^1 = x^*(t_1)$. Since x^* maximises J, the inequality $J(x^*) \geq J(y)$ holds in particular for all admissible functions $y(t)$ that satisfy $y(t_1) = y^1$. Therefore x^* is also a solution of the basic problem on the fixed time interval $[t_0, t_1]$ and with data x^0, y^1. Consequently x^* satisfies the Euler–Lagrange equation.

The transversality condition (i) can be proved as a continuation of the proof of Theorem 2: since the terminal value $x(t_1)$ is not fixed in this context, it is possible that the variation $\mu(t)$ is such that $\mu(t_1) \neq 0$ (but $\mu(t_0) = 0$ is still required); again it is seen that $I'(0) = 0$. Since it is already known that x^* solves the Euler–Lagrange equation, it follows from the integration by parts leading to (11) that

$$\mu(t_1) F_3'(t_1, x^*(t_1), \dot{x}^*(t_1)) = 0. \tag{24}$$

Taking μ such that $\mu(t_1) = 1$ the conclusion in (i) follows. With a little more effort also (ii) can be obtained along these lines. However, (iii) requires the implicit function theorem and a longer argument, so details are skipped here.

□

Sufficient conditions for solutions

Finally, a result on sufficient conditions for a solution is given. However, it holds only in case the basic function $F(t,x,u)$ is *concave* with respect to (x,u). This means that the Hessian matrix is negative semi-definite at all points, i.e. $(\frac{\partial^2 F}{\partial x \partial u}) \leq 0$; that is, for all t, x, u, this matrix has eigenvalues λ_1, λ_2 in $]-\infty, 0]$.

Theorem 10. *Suppose $F(t,x,u)$ is* concave *with respect to (x,u), and that $x^*(t)$ fulfils the Euler–Lagrange equation and one of the terminal conditions a) $x(t_1) = x^1$, b) $x(t_1) \geq x^1$, or c) $x(t_1)$ free, together with the corresponding transversality condition b) $F_3'(t_1, x^*(t_1), \dot{x}^*(t_1)) \leq 0$ (and $= 0$ if $x^*(t_1) > x^1$), respectively c) $F_3'(t_1, x^*(t_1), \dot{x}^*(t_1)) = 0$. Then x^* is a global maximiser, i.e. for every other admissible function $x(t)$ it holds true that*

$$\int_{t_0}^{t_1} F(t, x(t), \dot{x}(t))\, dt \leq \int_{t_0}^{t_1} F(t, x^*(t), \dot{x}^*(t))\, dt. \tag{25}$$

Since minimisation of $\int F(t,x,\dot{x})\,dt$ is achieved by maximising for $-F$, one has the analogous minimisation result for convex functions.

Example 11. Since $F(x,u) = x^2 + u^2$ is *convex* in Example 3, the solution $x^*(t) = e^{1+t} - e^{1-t}$ actually *minimises* the functional $J(x) = \int_0^1 (x(t)^2 + \dot{x}(t)^2)\,dt$ subject to the conditions $x(0) = 0$ and $x(1) = e^2 - 1$. (As was claimed earlier.)

Exercises

Exercise 1 (i) Calculate the value of $J(x) = \int_0^1 (x^2 + \dot{x}^2)\,dt$ in the cases
(a) $x(t) = (e^2 - 1)t$,
(b) $x(t) = e^{2t} - 1$,
(c) $x(t) = e^{1+t} - e^{1-t}$,
(d) $x(t) = at^2 + (e^2 - 1 - a)t$.
(They all go through $(0,0)$ and $(1, e^2 - 1)$, hence are admissible.)
(ii) Let $a \to \infty$ and conclude that $J(x)$ has no maximum on the curves joining $(0,0)$ to $(1, e^2 - 1)$.

Exercise 2 Let $J(x) = \int_1^2 t^{-2} \dot{x}(t)^2\,dt$ and consider the boundary conditions $x(1) = 1$, $x(2) = 2$.
(i) Find the admissible solutions to the Euler–Lagrange equation.
(ii) Show that the maximisation problem for J has no solution. (Try $x(t) = at^2(1 - 3a)t + 2a$.)
(iii) Does the above imply that the solution in (i) minimises $J(x)$?

Exercise 3 Consider the problem

$$\min \int_0^1 (t + x)^4\,dt, \qquad x(0) = 0, \quad x(1) = a.$$

Find the solution of the Euler–Lagrange equation and determine the value of a for which this solution is admissible. For this value of a, find the solution of the problem.

Exercise 4 The length of the graph of a C^1-function $x(t)$ which connects (t_0, x^0) to (t_1, x^1) is given by

$$L(x) = \int_{t_0}^{t_1} \sqrt{1 + \dot{x}(t)^2}\, dt.$$

Prove that $L(x)$ attains its minimum over the admissible functions exactly when $x(t)$ has a straight line as its graph.

Exercise 5 Let $A(t)$ denote the assets (or wealth) of a person at time t, let w be the constant wage, and suppose money can be borrowed at the fixed interest rate r; thus the consumption at time t is modelled as $C(t) = rA(t) + w - \dot{A}(t)$.

Suppose the person wants to maximise consumption from now until the expected death date T,

$$\int_0^T U(C(t))e^{-\rho t}\, dt.$$

Hereby U is a certain utility function, $U' > 0 > U''$, and ρ is a discount factor. While the present assests are A_0, the purpose is also to leave at least the amount A_T to the heirs, i.e. to have $A(T) \geq A_T$.

Apply the necessary conditions to this case. Show in particular that $A(t)$ is only optimal if $A(T) = A_T$. (Is this understandable?)

Solve Euler equation if $U(C) = a - e^{-bC}$ for constants a, $b > 0$.

Exercise 6 Investigate what the Euler–Lagrange equations give in the special cases when

- $F = F(t, x)$,
- $F = F(t, \dot{x})$,
- $F = F(x, \dot{x})$. Prove that here

$$\tfrac{d}{dt} F_3'(x, \dot{x}) = F_2'(x, \dot{x}) \tag{26}$$

$$\implies F(x, \dot{x}) - \dot{x} F_3'(x, \dot{x}) \text{ is constant} \tag{27}$$

$$\implies \dot{x} = 0 \quad \text{or} \quad \tfrac{d}{dt} F_3'(x, \dot{x}) = F_2'(x, \dot{x}). \tag{28}$$

PROBLEMS IN OPTIMAL CONTROL THEORY:

Exercise 1 Let $M = \left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$ and $N = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Find the controllability matrix G. Is the system $\dot{x} = Mx + Nu$ controllable ?

Is there an obvious reason for this ?

Exercise 2 Consider $M = \left(\begin{smallmatrix}0&\theta\\0&0\end{smallmatrix}\right)$ for $\theta > 0$; and $N = \left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$. Find the controllability matrix G. Is the system $\dot{x} = Mx + Nu$ controllable ?

Is the answer surprising in comparison with exercise 1 ?

Exercise 3 Suppose that $x^0 \in \mathcal{C}$, is it then also possible to pick the control $\alpha(\cdot)$ such that 0 is steered to x^0 in finite time ?

Exercise 4 For problems with $A = \mathbb{R}^m$, show that the controllable set $\mathcal{C}$ is a linear subspace of $\mathbb{R}^n$.

Show moreover that $\operatorname{rank} G = n \iff \mathcal{C} = \mathbb{R}^n$. (*Hint* $x \perp \mathcal{C}$ holds if and only if $G^T x = 0$.)

Exercise 5 Consider the railroad rocket car problem; i.e. Example 5 p.9–12.

(I) Write this problem down as a control problem, using our formalism: find $P(\alpha)$, the ODE and $\mathcal{A}$.
Recall why this system is controllable (*hint* p.22). What does Theorem 3.1 tell us about the problem ?

(II) Write down explicitly the consequences of Theorem 3.3, the Maximum Principle.

(III) Compute e^{tM} for this system, and find $h^T X(t)^{-1} N$.

(IV) Use $h^T X(t)^{-1} Na$ to show that the optimal control $\alpha^*(t)$ switches *once* between $+1$ and -1.
Was the solution sketched on p.10–12 correct ?

(V) Is the value of $h = (h_1, h_2)$ important ?

Exercise 6 Consider again the rocket car governed by

$$\begin{pmatrix}\dot{x}^1(t)\\ \dot{x}^2(t)\end{pmatrix} = \begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}x^1(t)\\ x^2(t)\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\alpha(t).$$

(I) Find the Hamiltonian function $H(x, p, a)$ as a function of $\left(\begin{smallmatrix}x^1\\x^2\end{smallmatrix}\right)$ and $\begin{pmatrix}p^1\\p^2\end{pmatrix}$ in $\mathbb{R}^2$; cf. Theorem 3.4.

(II) Write down the conclusions from Theorem 3.4.

(III) Show that $\dot{p}^*(t)^T = -p^*(t)^T M$ and determine $p^*(t)$.
Does this way of attack provide us with a natural candidate for the vector h in Theorem 3.3 ?

Exercise 7 Formulate the problem in calculus of variations as a control problem with $\alpha(t) = \dot{x}(t)$. Analyse it by means of Pontryagin's maximum principle, and deduce the Euler–Lagrange equation.

Exercise 8 Continue from the maximum principle in exercise 7 and derive Legendre's necessary conditions by inspecting the second order derivatives of H.

Exercise 9 As a simple example, apply the maximum principle to the problem of maximising $\int_0^1 x(t)\,dt$, when $\dot{x}(t) = x(t) + \alpha(t)$ and $x(0) = x^0$ with $x(1)$ free; hereby $-1 \le \alpha(t) \le 1$ for all t.

Show first that $p(t) = e^{1-t} - 1$, then that $\alpha^*(t) \equiv 1$. (Is this surprising ?)

Exercise 10 Consider an arbitrary control problem of maximising

$$\int_0^T r(t, x(t), \alpha(t))\, dt$$

subject to $\alpha(t) \in [0, \beta]$ and

$$\dot{x}(t) = \alpha(t), \quad x(0) = x^0, \quad x(T) \geq x^1.$$

Show that if $x^1 - x^0 = \beta T$ then $\alpha^*(t) \equiv \beta$ is the only solution.

What is the situation if $x^1 - x^0 > \beta T$?

What is "wrong" in this problem ?

Exercise 11 If the problem is to maximise $\int_0^1 (\alpha(t) - 2\beta(t))\, dt$ when $\dot{x} = (\alpha - \beta)^2$ and $x(0) = 0 = x(1)$ and both controls $\alpha(t)$ and $\beta(t)$ belong to $[-1, 1]$, show directly that any admissible control must satisfy $\alpha(t) = \beta(t)$. Deduce then that $\alpha^*(t) = \beta^*(t) \equiv -1$ is the solution.

Consequently the conclusions of Pontryagin's maximum principle are valid for these α^* and β^*. Show that this problem is *abnormal* in the sense that $p_0 = 0$.

Exercise 12 Consider the maximisation of $\int_0^1 (x(t) + \alpha(t))\, dt$ when $\dot{x} = -x + \alpha + t$, $x(0) = 1$ and $x(1)$ is free; $0 \leq \alpha \leq 1$. Use the maximum principle to find a unique candidate for p^*, u^* and x^*.

Confirm that you have found the solution. (*Hint:* The solution formula applies directly to the differential equation.)