The permanent process

August 24, 2005

Peter McCullagh and Jesper Møller

University of Chicago and Aalborg University

Abstract: We extend the boson process first to a large class of Cox processes and second an even larger class of infinitely divisible point processes. Density and moment results are studied in detail. These results are obtained in closed form as weighted permanents, so the extension is called a permanent process. Temporal extensions and a particularly tractable case of the permanent process are also studied. Extensions of the ferminon process along similar lines, leading to so-called determinant processes, are discussed at the end. While the permanent process is attractive, the determinant process is repulsive.

Keywords: boson process; Cox process; density of spatial point process; determinant process; factorial moment measure; fermion process; Gaussian process; infinite divisibility; Palm distribution; simulation; spatio-temporal process; weighted determinant; weighted permanent.

1 Introduction

Cox process models for spatial point processes form a rich class of models for aggregated point patterns, see e.g. Cox & Isham (1980), Stoyan et al. (1995), Daley & Vere-Jones (2003), Diggle (2003) and Møller & Waagepetersen (2003). In applications a spatial point pattern is observed within a bounded window $S \subset \mathbb{R}^d$, and a Cox process restricted to S is a finite random subset $\mathbf{X}$ of S whose distribution is usually specified indirectly by a non-negative random intensity function $\mathbf{\Lambda} = (\Lambda(x))_{x\in S}$ such that $\int_S \Lambda(x)\,\mathrm{d}x < \infty$ almost surely. Given the random intensity function, $\mathbf{X}$ is a Poisson process on S with intensity function $\mathbf{\Lambda}$, so the marginal density of the point process $\mathbf{X}$ is

$$f(\mathbf{x}) = \mathrm{E}\left[\exp\left(|S| - \int_S \Lambda(x)\,\mathrm{d}x\right)\prod_{j=1}^{n}\Lambda(x_j)\right] \tag{1}$$

for finite point configurations $\mathbf{x} = \{x_1, \ldots, x_n\} \subset S$ (see e.g. Møller & Waagepetersen, 2003). Here $n(\mathbf{x}) = n$, the number of points in $\mathbf{x}$, can be any non-negative

integer; the points in $\mathbf{x}$ are pairwise different; the density is with respect to the unit rate Poisson process on S; the expectation is with respect to $\mathbf{\Lambda}$; and $|S|$ is the volume of S.

For models so far considered in the literature, apart from simple cases such as a mixed Poisson process where $\mathbf{\Lambda}$ is the product of a positive random variable and a non-negative deterministic function, the expectation in (1) is computable only by Markov chain Monte Carlo (MCMC) methods. Examples are shot noise Cox processes (Brix 1999, Møller 2003) and log Gaussian Cox processes (Møller et al. 1998). Indeed, closed form expressions for densities of other kinds of spatial point process models are also unknown apart from trivial cases closely related to Poisson processes. For example, for the important class of Markov point processes (Van Lieshout 2000), the normalizing constant of the density cannot be evaluated explicitly.

The absence of a closed form for the density (1) has motivated us to study a class of flexible Cox process models with intensity function defined for any positive integer k and real covariance function by $C(x, x')$, $x, x' \in S$ by

$$\Lambda(x) = Z_1(x)^2 + \ldots + Z_k(x)^2 \tag{2}$$

where $\mathbf{Z}_j = (Z_j(x))_{x \in S}$, $j = 1, \ldots, k$ are independent zero-mean Gaussian processes with covariance function $C/2$. This class is similar in many respects to the log Gaussian Cox process, except that the density is expressed in terms of a weighted matrix permanent. Consequently such a point process $\mathbf{X}$ is called a *permanent process with parameters* $\alpha = k/2$ *and* C. The boson (or photon) process (Macchi 1971 and 1975, Grandell 1976, Daley and Vere-Jones 2003) corresponds to $\alpha = 1$. Another special case is a mixed Poisson process, i.e. when $C(x, x') = c$ is constant and hence $\Lambda(x) \sim (c/2)\chi^2(k)$ does not depend on $x \in S$.

Section 2 provides some moment properties of the permanent process and derives the density, assuming C has a spectral representation. These results are expressed in terms of sums of products of covariances. For any points $x_1, \ldots, x_n \in S$, the symbol $[C](x_1, \ldots, x_n)$ denotes the $n \times n$ matrix with entries $C(x_i, x_j)$. The key building block is the sum of cyclic products

$$\mathrm{cyp}[C](x_1, \ldots, x_n) = \sum_{\sigma : \#\sigma = 1} C(x_1, x_{\sigma(1)}) \cdots C(x_n, x_{\sigma(n)})$$

where σ is a permutation and $\#\sigma$ is the number of cycles. The α-weighted permanent

$$\mathrm{per}_\alpha[C](x_1, \ldots, x_n) = \sum_\sigma \alpha^{\#\sigma} C(x_1, x_{\sigma(1)}) \cdots C(x_n, x_{\sigma(n)})$$

is the sum over all permutations. This is a polynomial of degree n in α in which the coefficient of degree r is the sum of products over permutations having exactly r cycles. The usual permanent corresponds to $\alpha = 1$ (Minc 1978).

The density obtained in Section 2 requires 2α to be a positive integer. Section 3 shows that if certain weak conditions are satisfied (e.g. C need not be

a covariance function), the process exists for each $\alpha > 0$ and is infinitely divisible. The process is a Poisson randomization and there exists a non-trivial conditional limit process as $\alpha \to 0$ given that $\mathbf{X}$ is not empty. Furthermore, Section 3 establishes that the moment properties and the density of $\mathbf{X}$ are of a similar form as in Section 2, so we also call the extended process a permanent process.

Section 4 discusses temporal extensions of the permanent process. The special permanent process in which C is proportional to a projection is discussed in Section 5; it is called special on account of its striking and unusual properties. Section 6 concludes with a discussion on how to extend the fermion process along similar lines as for the boson process; the fermion process is a point process dual to the boson process (Bernard & Macchi 1973, Macchi 1975, Daley & Vere-Jones 2003). Thereby so-called determinant point processes are obtained. While the permanent process is attractive, the determinant process is repulsive.

Valiant (1979) has shown that exact computation of permanents of general matrices is a #P (sharp P) complete problem, so no deterministic polynomial-time algorithm is available. However, polynomial-time algorithms exist for certain special cases, such as general fixed-rank matrices (Barvinok 1996), and for approximate Monte-Carlo computation of general non-negative matrices (Jerrum et al. 2004). For most statistical purposes, approximate computation of permanent ratios is sufficient, which is a less demanding task. In addition, analytic approximations are available for large α. Statistical aspects and algorithms for calculating weighted permanents, permanent ratios and likelihoods will be discussed in more detail in future work.

2 The integer case of 2α

Throughout this section 2α is assumed to be a positive integer, so that $\mathbf{X}$ is the Cox process driven by (2). Extensions of the results are given in Section 3.

2.1 Moment properties

For a finite point process with density f with respect to the unit rate Poisson process Φ on S, the nth order product density is given by $\rho^{(n)}(x_1, \ldots, x_n) = \mathrm{E} f(\Phi \cup \{x_1, \ldots, x_n\})$ for any $n \in \mathbb{N}$ and pairwise different points $x_1, \ldots, x_n \in S$, see e.g. Daley & Vere-Jones (2003). Intuitively, $\rho^{(n)}(x_1, \ldots, x_n)\, \mathrm{d}x_1 \cdots \mathrm{d}x_n$ is the probability of observing n points from $\mathbf{X}$ occurring jointly in each of n infinitesimally small balls with centers $x_1, \ldots, x_n$ and volumes $\mathrm{d}x_1, \ldots, \mathrm{d}x_n$. Moreover, the nth order factorial moment measure has density $\rho^{(n)}$ with respect to Lebesgue measure on $\mathbb{R}^{dn}$.

Since $\mathbf{X}$ is a Cox process driven by (2), its nth order product density is given by

$$\rho^{(n)}(x_1, \ldots, x_n) = \mathrm{E}\left[\Lambda(x_1) \cdots \Lambda(x_n)\right]. \tag{3}$$

Theorem 1 below shows that this is a weighted permanent.

Lemma 1 Let $(Z(x))_{x\in S}$ be a zero-mean real Gaussian process with covariance function $C/2$. For any $x_1,\dots,x_n \in S$, not necessarily distinct, the joint cumulant of order n of the variables $Z(x_1)^2,\dots,Z(x_n)^2$ is

$$\mathrm{cum}_n(Z(x_1)^2,\dots,Z(x_n)^2) = \mathrm{cyp}[C](x_1,\dots,x_n)/2. \tag{4}$$

Proof: This is a standard application of the lattice-sum formula (Malyshev 1980, McCullagh 1984) for generalized cumulants as a sum of products of ordinary cumulants. All Gaussian cumulants are zero except those of order two, so the result is a sum of products of covariances, $C_{i_1,j_1}\cdots C_{i_n,j_n}$, where $C_{i,j} = C(x_i,x_j)$. Since each value $1,\dots,n$ occurs once as a first index and once as a second, $(j_1,\dots,j_n)$ is a permutation of $(i_1,\dots,i_n)$. Non-cyclic permutations do not satisfy the lattice connectivity condition, so the sum is restricted to cyclic permutations. For each cyclic permutation, there are 2^{n-1} distinct partitions of the $2n$ indices that satisfy the connectivity condition all giving rise to the same product $C_{1,\sigma(1)}\cdots C_{n,\sigma(n)}/2^n$. Consequently the joint cumulant is one half the sum of cyclic products.

Theorem 1 For any $x_1,\dots,x_n \in S$, not necessarily distinct,

$$\mathrm{E}\left[\Lambda(x_1)\cdots\Lambda(x_n)\right] = \mathrm{per}_\alpha[C](x_1,\dots,x_n). \tag{5}$$

Proof: Since $\Lambda(x)$ in (2) is the sum of k i.i.d. random processes, the joint cumulant of order n of $\Lambda(x_1),\dots,\Lambda(x_n)$ is k times the joint cumulant of $Z(x_1)^2,\dots,Z(x_n)^2$, which is given by (4). The joint moment of order n is the sum over sub-partitions of $\{1,\dots,n\}$ of cumulant products, one cumulant for each block of the partition. Since permutation cycles determine the blocks of the partition, the number of blocks is equal to the number of cycles. As a result, the term corresponding to the permutation σ has a factor $(k/2)^{\#\sigma}$, so the sum is the weighted permanent with weight $\alpha = k/2$.

In the complex version of Theorem 1, $Z_1,\dots,Z_k$ are independent zero-mean complex Gaussian processes with covariance function $\mathrm{cov}(Z_r(x),\bar{Z}_s(x')) = \delta_{rs}C(x,x')$, where C is Hermitian. Then the joint cumulant of $|Z_1(x_1)|^2,\dots,|Z_1(x_n)|^2$ in Lemma 1 is $\mathrm{cyp}[C](x_1,\dots,x_n)$, and the joint moment in Theorem 1 is $\mathrm{per}_k[C](x_1,\dots,x_n)$. The result for $\alpha = 1$ can be found in Macchi (1971, 1975).

By Theorem 1, $\rho^{(n)}(x_1,\dots,x_n) = \mathrm{per}_\alpha[C](x_1,\dots,x_n)$ for any pairwise distinct $x_1,\dots,x_n \in S$. In particular, $\rho(x) = \rho^{(1)}(x)$ is the intensity function and, provided $\rho(x)\rho(x') > 0$, $g(x,x') = \rho^{(2)}(x,x')/(\rho(x)\rho(x'))$ is the pair correlation function. Recall that for a Poisson process, $g = 1$.

Corollary 1 Let $\mathrm{cor}(x,x') = C(x,x')/(C(x,x)C(x',x'))^{1/2}$ be the correlation function (provided $C(x,x)C(x',x') > 0$). Then

$$\rho(x) = \alpha C(x,x), \quad g(x,x') = 1 + \mathrm{cor}(x,x')^2/\alpha. \tag{6}$$

Note that $g \geq 1$, in accordance with the usual interpretation that this indicates aggregation of the points in $\mathbf{X}$. In particular, $g \to 1$ as $\alpha \to \infty$, which

is to be expected, since $\mathbf{X}$ can be viewed as the superposition of k independent copies of the permanent process with parameter $k = 1$. In some sense the process becomes close to a Poisson process as $\alpha \to \infty$, since $\Lambda(x)/\alpha$ converges almost surely to $C(x, x)$.

Non-parametric estimation of the pair correlation function is based on kernel methods (Stoyan & Stoyan 1995), which may underestimate the support of $\mathrm{cor}(x, x')$ (particularly, in the extreme case $\mathrm{cor}(x, x') = 1$ of a mixed Poisson process). Non-parametric estimation of the inhomogenous K-function is less problematic (Baddeley et al. 2000): For the moment, assume that we extend the Cox process and the underlying Gaussian processes to $\mathbb{R}^d$. Clearly, the moment results above then apply for any points in $\mathbb{R}^d$. When the correlation function is stationary (i.e. when $\mathrm{cor}(x, x') = \mathrm{cor}(x - x')$ for any $x, x' \in \mathbb{R}^d$), the permament process is second order reweighted stationary and the inhomogenous K-function is given by

$$K_{\mathrm{inhom}}(t) = \int_{\|y\| \le t} g(y)\,\mathrm{d}y = \frac{\pi^{d/2}}{\Gamma(1 + d/2)} + \frac{1}{\alpha} \int_{\|y\| \le t} \mathrm{cor}(y)^2\,\mathrm{d}y, \quad t \ge 0.$$

If the covariance function is stationary (i.e. when $C(x, x') = C(x - x')$ for any $x, x' \in \mathbb{R}^d$), the permanent process is stationary (i.e. its distribution is invariant under translations in $\mathbb{R}^d$), so $\rho(x) = \rho$ is constant, $g(x, x') = g(x - x')$ and the inhomogeneous K-function agrees with Ripley's K-function (Ripley 1976 and 1977).

In the case of a log Gaussian Cox process, i.e. if $\log \mathbf{\Lambda}$ in (1) is a real Gaussian process with mean function μ and covariance function C, we have a simpler result for the product density (Møller et al. 1998):

$$\rho^{(n)}(x_1, \ldots, x_n) = \exp\left(\sum_{i=1}^{n} \mu(x_i) + \sum_{i,j=1}^{n} C(x_i, x_j)/2 \right).$$

Unlike the permanent process, however, the density for the log Gaussian Cox process is not available in closed form: see Section 2.3.

2.2 Conditions

In Section 2.3 and also some times later on we make the following assumptions.

Equip the space $\mathcal{L}_2(S)$ of square integrable real Borel functions on S with the usual inner product $\langle p, q \rangle = \int_S p(x)q(x)\,\mathrm{d}x$ and norm $\|p\|_2 = \langle p, p \rangle^{1/2}$. Suppose that the covariance function has spectral representation

$$C(x, x') = \sum_{r=0}^{\infty} \lambda_r e_r(x) e_r(x'), \quad x, x' \in S \tag{7}$$

where the e_r form an orthonormal basis of $\mathcal{L}_2(S)$ and the eigenvalues λ_r are non-negative. Indeed a spectral representation holds for most covariance functions: since $\mathcal{L}_2(S)$ is a separable Hilbert space, (7) holds with respect to some

orthonormal basis if and only if C is a compact operator (meaning that for any bounded sequence $\{q_r\} \subset \mathcal{L}_2(S)$, $\{Cq_r\}$ has a subsequence convergent in $\mathcal{L}_2(S)$ where $Cq_r(x) = \int_S C(x,x')q_r(x')\,\mathrm{d}x'$). For instance, C is compact if it is continuous and S is compact. See Reed and Simon (1980).

We take

$$Z_j(x) = \sum_{r=0}^{\infty} V_{j,r} e_r(x)$$

where the $V_{j,r}$ are independent $N(0, \lambda_r/2)$-distributed random variables. Clearly, $Z_1, \ldots, Z_k$ are then independent zero-mean Gaussian processes with covariance function $C/2$. Assuming

$$\sum_{r=0}^{\infty} \lambda_r < \infty \tag{8}$$

then

$$\int_S \Lambda(x)\,\mathrm{d}x = \sum_{j=1}^{k} \sum_{r=0}^{\infty} V_{j,r}^2 \tag{9}$$

is almost surely finite. Condition (8) means that

$$\mathrm{E}n(\mathbf{X}) = \mathrm{E}\int_S \Lambda(x)\,\mathrm{d}x = \alpha \sum_{r=0}^{\infty} \lambda_r$$

is finite.

The rank of C is the number of non-zero eigenvalues. Note that C is a projection if and only if each λ_r is either one or zero, and (8) implies then that $\mathrm{rank}(C) < \infty$. For C proportional to a projection of rank one, the permanent process is a mixed Poisson process.

For functions $h : S^n \mapsto [0, \infty)$, we define

$$\int_{S^n} h(\mathbf{x})\,\mathrm{d}\mathbf{x} = \int_S \cdots \int_S h(x_1, \ldots, x_n)\,\mathrm{d}x_1 \cdots \mathrm{d}x_n$$

where for $n = 0$, we interpret the integral as $h(\emptyset)$, where $\emptyset$ is the empty point configuration. Further, if $h(x_1, \ldots, x_n)$ is a symmetric function, we do not distinguish between whether $\mathbf{x}$ is interpreted as a vector or a point configuration. Indeed product densities, weighted permanents and many other functions considered in the sequel are symmetric functions, and it is often convenient just to write $\rho^{(n)}(\mathbf{x})$, $\mathrm{per}_\alpha[C](\mathbf{x})$, $\mathrm{cyp}[C](\mathbf{x})$ and so on for a finite point configuration $\mathbf{x} \subset S$ of $n(\mathbf{x}) = n$ points. Finally, we set

$$\rho^{(0)}(\emptyset) = \mathrm{per}_\alpha[C](\emptyset) = 1, \quad \mathrm{cyp}[C](\emptyset) = 0.$$

2.3 Density function

Let the situation be as in Section 2.2. In order to derive the density of the permanent process it is convenient to introduce $\tilde{C}$, $\tilde{\mathbf{\Lambda}} = (\tilde{\Lambda}(x))_{x \in S}$, $\tilde{\mathbf{X}}$ and $\tilde{\rho}^{(n)}$

in the same way as above for C, $\mathbf{\Lambda} = (\Lambda(x))_{x\in S}$, $\mathbf{X}$ and $\rho^{(n)}$, except that we replace the eigenvalues by

$$\tilde{\lambda}_r = \lambda_r/(1+\lambda_r), \quad r = 0, 1, \ldots. \tag{10}$$

Everything is again well-defined, since $\sum_0^\infty \tilde{\lambda}_r \le \sum_0^\infty \lambda_r < \infty$. Furthermore, define

$$D = \sum_{r=0}^{\infty} \log(1+\lambda_r) = -\sum_{r=0}^{\infty} \log(1-\tilde{\lambda}_r).$$

Theorem 2 The density of the permanent process $\mathbf{X}$ at any finite point configuration $\mathbf{x} \subset S$ is

$$f(\mathbf{x}) = \mathrm{e}^{|S|-\alpha D} \operatorname{per}_\alpha[\tilde{C}](\mathbf{x}). \tag{11}$$

Proof: Combining (1) and (9),

$$f(\mathbf{x}) = \mathrm{e}^{|S|} \mathrm{E}\left[\left(\prod_{j=1}^{k}\prod_{r=0}^{\infty} \mathrm{e}^{-V_{j,r}^2}\right)\left(\prod_{j=1}^{n}\Lambda(x_j)\right)\right]$$

where $V_{j,r} \sim \mathrm{N}(0, \lambda_r/2)$, $j = 1, \ldots, k$, $r = 0, 1, \ldots$ are independent. If g_r is the density of $\mathrm{N}(0, \lambda_r/2)$ and $\tilde{g}_r$ is the density of $\mathrm{N}(0, \tilde{\lambda}_r/2)$, then $\exp\left(-v^2\right) g_r(v) = \tilde{g}_r(v)/(1+\lambda_r)^{1/2}$. Consequently,

$$\mathrm{E}\left[\left(\prod_{j=1}^{k}\prod_{r=0}^{\infty} \mathrm{e}^{-V_{j,r}^2}\right)\left(\prod_{j=1}^{n}\Lambda(x_j)\right)\right] = \mathrm{E}\left[\prod_{j=1}^{n}\tilde{\Lambda}(x_j)\right]\prod_{r=0}^{\infty}(1+\lambda_r)^{-\alpha}.$$

Hence, by (5), we obtain (11).

Theorem 2 was first established for the special case of the boson process ($\alpha = 1$) by Macchi (1971, 1975) who noted that $\tilde{C}$ is the root of the integral equation

$$\tilde{C}(x,x') + \int_S \tilde{C}(x,y)C(y,x')\,\mathrm{d}y = C(x,x').$$

See also Grandell (1976) and Daley & Vere-Jones (2003).

For $n(\mathbf{x}) = n$,

$$f(\mathbf{x}) \propto \tilde{\rho}^{(n)}(\mathbf{x}) \tag{12}$$

where the constant of proportionality is $\exp(|S| - \alpha D)$. Here is an example of a special kind of shot noise Cox process with a similar property: consider a Cox process driven by $\Lambda(x) = \sum \Gamma_r q_r(x)$, where q_r is a kernel (density function) on S and the Γ_r are independent gamma distributed random variables with shape parameter $\alpha_r > 0$ and scale parameter $\lambda_r \ge 0$ such that $\sum \alpha_r\lambda_r < \infty$. Let $\tilde{\rho}^{(n)}$ be the nth order product density of the similar process when the λ_r are replaced by the $\tilde{\lambda}_r$. Then (12) is still true; the proof is analogous to that of Theorem 2. While Theorem 1 applies for the permanent process, we have not found any

useful way of computing $\tilde{\rho}^{(n)}$ for the shot noise Cox process (unless n is very small).

Given that a finite point pattern $\mathbf{x} \subset S$ is observed and $f(\mathbf{x}) > 0$, the conditional mean of $\Lambda(x)$ is the Bayes estimate of the intensity at $x \in S \setminus \mathbf{x}$: For any Cox process with density (1), it can be straightforwardly verified that

$$\mathrm{E}\big(\Lambda(x)\big|\,\mathbf{x}\big) = f(\mathbf{x} \cup \{x\})/f(\mathbf{x}) \tag{13}$$

where the expression on the right side is the Papangelou conditional intensity (see e.g. Møller & Waagepetersen 2003). Since the density of the permanent process at $\mathbf{x}$ is proportional to $\mathrm{per}_\alpha[\tilde{C}](\mathbf{x})$, the Bayes estimate for the intensity is the α-weighted permanent ratio

$$\mathrm{E}\big(\Lambda(x)\big|\,\mathbf{x}\big) = \frac{\mathrm{per}_\alpha[\tilde{C}](\mathbf{x} \cup \{x\})}{\mathrm{per}_\alpha[\tilde{C}](\mathbf{x})}. \tag{14}$$

3 Extensions

In the remainder of this paper, unless otherwise stated, we relax the conditions in Section 2.2 as follows.

Denote by $\mathbb{R}_0$ the set of non-negative real numbers. We assume that $\alpha \in \mathbb{R}_0$ and $\tilde{C} : S^2 \mapsto \mathbb{R}_0$ is symmetric with spectral representation

$$\tilde{C}(x, x') = \sum_{r=0}^{\infty} \tilde{\lambda}_r e_r(x) e_r(x'), \tag{15}$$

norm $\|\tilde{C}\| = \sup\{|\tilde{\lambda}_r|\} < 1$ and finite sum $\sum_0^\infty |\tilde{\lambda}_r| < \infty$. It is convenient here to re-define C in the form of the power series $C = \sum_{r=1}^\infty \tilde{C}^r$, where $\tilde{C}^r(x, x') = \sum_{j=0}^\infty \tilde{\lambda}_j^r e_j(x) e_j(x')$. This series is convergent and coincides with the spectral representation (7) where $\lambda_r = \tilde{\lambda}_r/(1 - \tilde{\lambda}_r)$. This follows from the facts that a) $\|\tilde{C}\| < 1$ guarantees that the sum $\sum_1^\infty \tilde{C}^r(x, x')$ converges and b) absolute summability ensures that

$$\begin{aligned} C(x, x') &= \sum_{r=1}^{\infty} \tilde{C}^r(x, x') = \sum_{r=1}^{\infty} \sum_{j=0}^{\infty} \tilde{\lambda}_j^r e_j(x) e_j(x') = \sum_{j=0}^{\infty} \sum_{r=1}^{\infty} \tilde{\lambda}_j^r e_j(x) e_j(x') \\ &= \sum_{j=0}^{\infty} \tilde{\lambda}_j/(1 - \tilde{\lambda}_j) e_j(x) e_j(x') = \sum_{j=0}^{\infty} \lambda_j e_j(x) e_j(x'). \end{aligned}$$

Observe also that C is non-negative, since each term $\tilde{C}^r$ is non-negative.

By admitting negative eigenvalues of C and non-integer values of 2α, the connection with (2) is severed, so it appears unlikely that the extensions of the permanent process to be established below are Cox processes in general.

3.1 Density of the extended process

Theorem 3 For each $\alpha \in \mathbb{R}_0$, there exists a point process with density (11).

Proof: For integers $n \geq 1$,

$$\begin{aligned}\int_{S^n} \mathrm{cyp}[\tilde{C}](\mathbf{x})\,\mathrm{d}\mathbf{x} &= \sum_{\sigma:\,\#\sigma=1} \sum_{r_1,\ldots,r_n=0}^{\infty} \int_{S^n} \prod_{j=1}^{n} \tilde{\lambda}_{r_j} e_{r_j}(x_j) e_{r_j}(x_{\sigma(j)})\,\mathrm{d}\mathbf{x} \\ &= (n-1)! \sum_{r_1,\ldots,r_n=0}^{\infty} \int_S \tilde{\lambda}_{r_1} e_{r_1}(x_1) e_{r_n}(x_1)\,\mathrm{d}x_1 \\ &\quad \int_S \tilde{\lambda}_{r_2} e_{r_1}(x_2) e_{r_2}(x_2)\,\mathrm{d}x_2 \cdots \int_S \tilde{\lambda}_{r_n} e_{r_{n-1}}(x_n) e_{r_n}(x_n)\,\mathrm{d}x_n \\ &= (n-1)! \sum_{r=0}^{\infty} \tilde{\lambda}_r^n \end{aligned} \tag{16}$$

where the second identity follows from Fubini's theorem, since each cyclic product has the same integral. Furthermore, with the convention that $\mathrm{cyp}[\tilde{C}](\emptyset) = 0$,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{S^n} \mathrm{cyp}[\tilde{C}](\mathbf{x})\,\mathrm{d}\mathbf{x} = \sum_{n=1}^{\infty} \sum_{r=0}^{\infty} \tilde{\lambda}_r^n / n = -\sum_{r=0}^{\infty} \log(1-\tilde{\lambda}_r) = D \tag{17}$$

where the reversal of the order of summation requires the eigenvalues to be absolutely summable and less than one in absolute value. For integers $r \geq 0$, define $\mathrm{cyp}^{(r)}[\tilde{C}](\mathbf{x})$ to be the sum of products over permutations having exactly r cycles, with the convention $\mathrm{cyp}^{(r)}[\tilde{C}](\emptyset) = 1$ and $\mathrm{cyp}^{(r)}[\tilde{C}](\mathbf{x}) = 0$ if either $r > n(\mathbf{x})$ or $r = 0 < n(\mathbf{x})$. Then

$$\begin{aligned}\frac{D^r}{r!} &= \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} \frac{1}{r! s_1! \cdots s_r!} \int_{S^{s_1}} \cdots \int_{S^{s_r}} \mathrm{cyp}[\tilde{C}](\mathbf{x}_1) \cdots \mathrm{cyp}[\tilde{C}](\mathbf{x}_r)\,\mathrm{d}\mathbf{x}_1 \cdots \mathrm{d}\mathbf{x}_r \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{S^n} \mathrm{cyp}^{(r)}[\tilde{C}](\mathbf{x})\,\mathrm{d}\mathbf{x}. \end{aligned}$$

Thus for any real $\alpha > 0$, since $\mathrm{per}_\alpha[\tilde{C}](\mathbf{x}) = \sum \alpha^r \mathrm{cyp}^{(r)}[\tilde{C}](\mathbf{x})$,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{S^n} \mathrm{per}_\alpha[\tilde{C}](\mathbf{x})\,\mathrm{d}\mathbf{x} = \mathrm{e}^{\alpha D}$$

and this identity also holds for $\alpha = 0$. Finally, for all $\alpha \geq 0$, $\mathrm{per}_\alpha[\tilde{C}](\mathbf{x}) \geq 0$, since $\tilde{C}(x, x') \geq 0$.

Henceforth, for any $\alpha \in \mathbb{R}_0$, $\mathbf{X}$ or $\mathbf{X}_\alpha$ denotes the permanent process as in Theorem 3 and f or f_α is its density:

$$f_\alpha(\mathbf{x}) = \mathrm{e}^{|S| - \alpha D} \mathrm{per}_\alpha[\tilde{C}](\mathbf{x}). \tag{18}$$

Note that $\mathbf{X}_\alpha = \emptyset$ almost surely if $\alpha = 0$ or $\tilde{C} \equiv 0$.

3.2 Moment properties

We now show that the product density for any $\alpha \in \mathbb{R}_0$ is given by $\mathrm{per}_\alpha[C](\mathbf{x})$. The first step in the proof is the following lemma.

Lemma 2 Let $\mathbf{x} = (x_1, \ldots, x_m)$ be a point in S^m where $m \geq 1$. Then

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{S^n} \mathrm{cyp}[\tilde{C}](\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{y} = \mathrm{cyp}[C](\mathbf{x}).$$

Note that for $m = 0$, i.e. $\mathbf{x}$ empty, the sum is D, cf. (17), and $D \neq \mathrm{cyp}[C](\emptyset)$ if $C \not\equiv 0$.

Proof: The main difficulty in the proof is notational. For positive integers $r_1, \ldots, r_m$, define the asymmetric cyclic product

$$[\tilde{C}^{r_1}, \ldots, \tilde{C}^{r_m}](\mathbf{x}) = \tilde{C}^{r_1}(x_1, x_2)\tilde{C}^{r_2}(x_2, x_3) \cdots \tilde{C}^{r_m}(x_m, x_1).$$

Consider a typical term in the expansion of $\mathrm{cyp}[\tilde{C}](\mathbf{x}, y_1, \ldots, y_n)$, initially keeping $m = 2$ for simplicity of exposition. A cyclic permutation of $(\mathbf{x}, \mathbf{y})$ beginning from the position of x_1 is followed by $r_1 \geq 0$ ys, then x_2 and the remaining $r_2 = n - r_1$ ys. The corresponding integral is

$$\begin{aligned}
&\int_S \cdots \int_S \tilde{C}(x_1, y_1)\tilde{C}(y_1, y_2) \cdots \tilde{C}(y_{r_1} - 1, y_{r_1})\tilde{C}(y_1, x_2) \\
&\tilde{C}(x_2, y_{r_1+1})\tilde{C}(y_{r_1+1}, y_{r_2+2}) \cdots \tilde{C}(y_{n-1}, y_n)\tilde{C}(y_n, x_1)\, \mathrm{d}y_1 \cdots \mathrm{d}y_n \\
&= [\tilde{C}^{r_1+1}, \tilde{C}^{r_2+1}](x_1, x_2).
\end{aligned}$$

Since there are $n!$ orders for the components of $\mathbf{y}$, we find that

$$\frac{1}{n!} \int_{S^n} \mathrm{cyp}[\tilde{C}](x_1, x_2, \mathbf{y})\, \mathrm{d}\mathbf{y} = \sum_{r=0}^{n} [\tilde{C}^{r+1}, \tilde{C}^{n-r+1}](x_1, x_2).$$

By reversal of the order of summation,

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{1}{n!} \int_{S^n} \mathrm{cyp}[\tilde{C}](x_1, x_2, \mathbf{y})\, \mathrm{d}\mathbf{y} = \sum_{n=0}^{\infty} \sum_{r=0}^{n} [\tilde{C}^{r+1}, \tilde{C}^{n-r+1}](x_1, x_2) \\
&= \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} [\tilde{C}^{r+1}, \tilde{C}^{n-r+1}](x_1, x_2) = \sum_{r=0}^{\infty} [\tilde{C}^{r+1}, C](x_1, x_2) = [C, C](x_1, x_2).
\end{aligned}$$

The same argument for general $m \geq 1$ gives

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{S^n} \mathrm{cyp}[\tilde{C}](\mathbf{x}, \mathbf{y})\, \mathrm{d}\mathbf{y} = \sum_{\sigma:\, \#\sigma = 1} [C, \ldots, C](\mathbf{x}_\sigma) = \mathrm{cyp}[C](\mathbf{x})$$

where $\mathbf{x}_\sigma = (x_{\sigma(1)}, \ldots, x_{\sigma(m)})$. This establishes the required result.

Theorem 4 For any finite point configuration $\mathbf{x} \subset S$ with $n(\mathbf{x}) = m \geq 1$,

$$\rho^{(m)}(\mathbf{x}) = \mathrm{per}_\alpha[C](\mathbf{x}). \tag{19}$$

Proof: For any integer $s \in \{1, \ldots, m\}$, recalling the definition of $\rho^{(m)}$ from Section 2.1,

$$\begin{aligned}\rho^{(m)}(\mathbf{x}) = \mathrm{E} f_\alpha(\Phi \cup \mathbf{x}) &= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\alpha D}}{n!} \int_{S^n} \mathrm{per}_\alpha[\tilde{C}](\mathbf{x} \cup \mathbf{y})\, \mathrm{d}\mathbf{y} \\ &= \sum_{r=s}^{\infty} \frac{\alpha^r}{(r-s)!} \mathrm{e}^{-\alpha D} \sum_{\mathbf{x}_1, \ldots, \mathbf{x}_r} \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} \frac{1}{s_1! \cdots s_r!} \\ &\int_{S^{s_1}} \cdots \int_{S^{s_r}} \mathrm{cyp}[\tilde{C}](\mathbf{x}_1 \cup \mathbf{y}_1) \cdots \mathrm{cyp}[\tilde{C}](\mathbf{x}_r \cup \mathbf{y}_r)\, \mathrm{d}\mathbf{y}_1 \cdots \mathrm{d}\mathbf{y}_r\end{aligned}$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_r$ is any partition of $\mathbf{x}$ into r blocks of which s are non-empty. The factor $(r-s)!$ arises from permutation of blocks containing no xs. Application of Lemma 2 gives

$$\rho^{(m)}(\mathbf{x}) = \frac{1}{m} \sum_{s=1}^{m} \mathrm{e}^{-\alpha D} \sum_{r=s}^{\infty} \frac{\alpha^r}{(r-s)!} \mathrm{cyp}[C](\mathbf{x}_1) \cdots \mathrm{cyp}[C](\mathbf{x}_s) D^{r-s}$$

the factor D^{r-s} arising from those blocks in the partition of $\mathbf{x} \cup \mathbf{y}$ that contain no xs. Simplification reduces this to

$$\sum_{s=1}^{m} \mathrm{e}^{-\alpha D} \sum_{r=s}^{\infty} \frac{\alpha^{r-s} D^{r-s}}{(r-s)!} \alpha^s \mathrm{cyp}^{(s)}[C](\mathbf{x}) = \mathrm{per}_\alpha[C](\mathbf{x}).$$

which is the required result (19).

Thus, for any $\alpha \in \mathbb{R}_0$, we obtain (6), noticing that $\mathrm{cor}(x, x')$ has only the interpretation of a correlation function when the eigenvalues λ_r are all non-negative. As α varies from 0 to ∞, the pair correlation function $g(x, x')$ decreases from arbitrary large values to one (provided $g(x, x')$ exists, i.e. $C(x, x) > 0$ and $C(x', x') > 0$). This indicates again that the degree of aggregation is a decreasing function of α, cf. Section 2.1. Moreover, if the eigenvalues λ_r are non-negative, $g(x, x') \leq 1 + 1/\alpha$.

3.3 Infinite divisibility and simulation

Infinite divisibility of f_α means that for each integer $n \geq 1$, if $\mathbf{Y}_1, \ldots, \mathbf{Y}_n$ are i.i.d. permanent processes with density $f_{\alpha/n}$, the superposition $\cup \mathbf{Y}_j$ is a permanent process with density f_α. To establish this we first establish the algebraic semi-group property of weighted permanents.

Lemma 3 For any reals α, α' and finite point configuration $\mathbf{x} \subset S$,

$$\mathrm{per}_{\alpha+\alpha'}[\tilde{C}](\mathbf{x}) = \sum_{\mathbf{w} \subseteq \mathbf{x}} \mathrm{per}_\alpha[\tilde{C}](\mathbf{w})\ \mathrm{per}_{\alpha'}[\tilde{C}](\mathbf{x} \setminus \mathbf{w}). \tag{20}$$

Proof: The claim is trivially true if $\mathbf{x}$ is empty. Assume then that $\mathbf{x}$ contains $n > 0$ points. For $\mathbf{w} \subseteq \mathbf{x}$, denote the complement $\bar{\mathbf{w}} = \mathbf{x} \setminus \mathbf{w}$, let m be the number of points in $\mathbf{w}$, and let σ_1 and σ_2 be permutations of $1, \ldots, m$ and $1, \ldots, n-m$, respectively, i.e. corresponding to the points in $\mathbf{w} = \{w_1, \ldots, w_m\}$ and $\bar{\mathbf{w}} = \{\bar{w}_1, \ldots, \bar{w}_{n-m}\}$. Further, for a given permutation σ of $1, \ldots, n$, the symbol $\mathbf{w}|\bar{\mathbf{w}} \geq \sigma$ denotes a partition in which $\mathbf{w}$ corresponds to a union of cycles of σ. In this context, σ_1 and σ_2 correspond to the restriction of σ to $\mathbf{w}$ and $\bar{\mathbf{w}}$, respectively, and $\#\sigma = \#\sigma_1 + \#\sigma_2$ is the number of cycles. With this notation,

$$
\begin{aligned}
&\sum_{\mathbf{w}\colon \mathbf{w}\subseteq\mathbf{x}} \mathrm{per}_\alpha[\tilde{C}](\mathbf{w})\,\mathrm{per}_{\alpha'}[\tilde{C}](\bar{\mathbf{w}}) \\
&= \sum_{\mathbf{w}\colon \mathbf{w}\subseteq\mathbf{x}} \sum_{\sigma_1}\sum_{\sigma_2} \alpha^{\#\sigma_1}\alpha'^{\#\sigma_2}\tilde{C}(w_1, w_{\sigma_1(1)})\cdots\tilde{C}(w_m, w_{\sigma_1(m)}) \\
&\qquad\qquad \tilde{C}(\bar{w}_1, \bar{w}_{\sigma_2(1)})\cdots\tilde{C}(\bar{w}_{n-m}, w_{\sigma_2(n-m)}) \\
&= \sum_{\sigma} \sum_{\mathbf{w}\colon \mathbf{w}\subseteq\mathbf{x},\, \mathbf{w}|\bar{\mathbf{w}}\geq\sigma} \frac{n!}{m!(n-m)!}\alpha^{\#\sigma_1}\alpha'^{\#\sigma_2}\,\tilde{C}(x_1, x_{\sigma(1)})\cdots\tilde{C}(x_n, x_{\sigma(n)}) \\
&= \sum_{\sigma} (\alpha+\alpha')^{\#\sigma}\,\tilde{C}(x_1, x_{\sigma(1)})\cdots\tilde{C}(x_n, x_{\sigma(n)})
\end{aligned}
$$

which is the required result.

Combining Lemma 3 with the density (18), we see that if $\mathbf{X}_\alpha$ and $\mathbf{X}_{\alpha'}$ are independent permanent processes, the superposition $\mathbf{X}_\alpha \cup \mathbf{X}_{\alpha'}$ is a permanent process with density $f_{\alpha+\alpha'}$. Consequently, since f_α exists for all $\alpha \in \mathbb{R}_0$, the permanent process is infinitely divisible.

In particular this implies infinite divisibility of the number of points $N = n(X)$, with cumulant generating function

$$
\begin{aligned}
\log \mathrm{E}\left[\mathrm{e}^{tN}\right] &= -\alpha D + \log\left(\sum_{n=0}^{\infty}\frac{\mathrm{e}^{tn}}{n!}\int_{S^n}\mathrm{per}_\alpha[\tilde{C}](\mathbf{x})\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right) \\
&= -\alpha D + \log\left(\sum_{n=0}^{\infty}\frac{1}{n!}\int_{S^n}\mathrm{per}_\alpha[\mathrm{e}^t\tilde{C}](\mathbf{x})\,\mathrm{d}x_1\cdots\mathrm{d}x_n\right) \\
&= -\alpha D - \alpha\sum_{r=0}^{\infty}\log(1-\mathrm{e}^t\tilde{\lambda}_r), \quad t \leq -\log\|\tilde{C}\|. \qquad (21)
\end{aligned}
$$

Thus N is distributed as the sum of independent negative binomial random variables with probability density functions

$$
\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)n!}\tilde{\lambda}_r^n(1-\tilde{\lambda}_r)^\alpha, \quad n = 0, 1, \ldots,
$$

for $r = 0, 1, \ldots$ (taking $0^0 = 1$). Consequently, N is over-dispersed and

$$
\mathrm{E}N = \alpha\sum_{r=0}^{\infty}\frac{\tilde{\lambda}_r}{1-\tilde{\lambda}_r} = \alpha\sum_{r=0}^{\infty}\lambda_r\,, \quad \mathrm{Var}N = \alpha\sum_{r=0}^{\infty}\frac{\tilde{\lambda}_r}{(1-\tilde{\lambda}_r)^2} = \alpha\sum_{r=0}^{\infty}\lambda_r(1+\lambda_r).
$$

The probability that $\mathbf{X}_\alpha$ is empty is $\exp(-\alpha D)$, which tends to one as $\alpha \to 0$. If we consider the conditional distributions given $\mathbf{X}_\alpha \neq \emptyset$, we get a non-trivial limit density $f_0(\mathbf{x}|\text{not } \emptyset)$ as $\alpha \to 0$, and we let W denote the number of points in this limiting process.

Corollary 2 For non-empty and finite point configurations $\mathbf{x} \subset S$,

$$f_0(\mathbf{x}|\text{not } \emptyset) = \mathrm{e}^{|S|} \operatorname{cyp}[\tilde{C}](\mathbf{x})/D \tag{22}$$

and

$$\mathrm{P}(W = n) = D^{-1} \sum_{r=0}^{\infty} \tilde{\lambda}_r^n / n, \quad n = 1, 2, \ldots. \tag{23}$$

Moreover, $\mathbf{X}_\alpha$ is a Poisson randomization, that is, if R is a Poisson distributed random variable with mean αD, then $\mathbf{X}_\alpha$ is distributed as the superposition of R i.i.d. point processes with density $f_0(\mathbf{x}|\text{not } \emptyset)$.

Proof: Equation (17) shows that $f_0(\mathbf{x}|\text{not } \emptyset)$ is a density with respect to the unit rate Poisson process on S. From Theorem 3 when $\alpha > 0$, the conditional density given that $\mathbf{X}_\alpha$ is non-empty is

$$\mathrm{e}^{|S|} \operatorname{per}_\alpha[\tilde{C}](\mathbf{x}) / \big(1 - \exp(-\alpha D)\big)$$

which tends to (22) as $\alpha \to 0$. Combining (16) and (22) we obtain (23). Finally, by the same arguments as in the proof of Theorem 3,

$$\begin{aligned}
\mathrm{P}(\mathbf{X} \in F) &= \sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\alpha D}}{n!} \int_{S^n} \mathbf{1}[\mathbf{x} \in F] \operatorname{per}_\alpha[\tilde{C}](\mathbf{x}) \, \mathrm{d}\mathbf{x} \\
&= \sum_{r=0}^{\infty} \frac{(\alpha D)^r \mathrm{e}^{-\alpha D}}{r!} \sum_{s_1=0}^{\infty} \cdots \sum_{s_r=0}^{\infty} \frac{\mathrm{e}^{-|S|}}{s_1!} \cdots \frac{\mathrm{e}^{-|S|}}{s_r!} \int_{S^{s_1}} \cdots \int_{S^{s_r}} \\
&\qquad \mathbf{1}[\mathbf{x}_1 \cup \ldots \cup \mathbf{x}_r \in F] f_0(\mathbf{x}_1|\text{not } \emptyset) \cdots f_0(\mathbf{x}_r|\text{not } \emptyset) \, \mathrm{d}\mathbf{x}_1 \cdots \mathrm{d}\mathbf{x}_r
\end{aligned}$$

from which the Possion randomization follows.

From (23) we obtain the expected number of points

$$\mathrm{E}W = D^{-1} \sum_{r=0}^{\infty} \tilde{\lambda}_r / (1 - \tilde{\lambda}_r) = \sum_{r=0}^{\infty} \lambda_r / D.$$

As an example of such a limit process, let $S = [0, 2\pi]$ and $\tilde{C}(x, x') = \theta(1 - \cos(x - x'))/(2\pi)$ for $0 < \theta < 1$, so $\tilde{C}(x, x) \geq 0$. The non-zero eigenvalues of $\tilde{C}$ are θ and $-\theta/2$ with multiplicities one and two, respectively. Note that $\tilde{C}$ is not positive semi-definite, and $\mathrm{E}W$ is an increasing function of θ with range $(0, \infty)$.

We can simulate from $f(\mathbf{x}|\text{not } \emptyset)$ by generating first $W = n$ and second the n points in W with conditional density

$$f_n(x_1, \ldots, x_n|\text{not } \emptyset) \propto \tilde{C}(x_1, x_2)\tilde{C}(x_2, x_3) \cdots \tilde{C}(x_n, x_1).$$

Gibbs sampling or another Metropolis-Hastings algorithm may easily work for simulating from the "full conditionals" with densities

$$\pi(x_i|\cdots) \propto \tilde{C}(x_{i-1}, x_i)\tilde{C}(x_i, x_{i+1})$$

where $x_{n+1} = x_1$.

This together with the Poisson randomization in Corollary 3 provide one way of simulating from f_α. Alternatively, if 2α is a positive integer, we may exploit the doubly-stochastic construction of the Cox process $\mathbf{X}_\alpha$, so that we first simulate the Gaussian processes and second the Poisson process $\mathbf{X}_\alpha|\mathbf{\Lambda}$.

4 Two temporal extensions

Spatial birth-death processes satisfying a detailed balance condition with respect to f_α can easily be constructed, when a birth is the addition of a single point and a death is the deletion of a single point (Ripley 1977, Møller & Waagepetersen 2003). However, the detailed balance condition requires the evaluation of the Papangelou conditional intensity (13). Below we consider two other spatio-temporal constructions.

4.1 An accretion process with independent increments

The Poisson randomization established in Corollary 2 implies there exists a coupling construction of the permanent processes $\mathbf{X}_\alpha$ for all $\alpha \in \mathbb{R}_0$. Interpreting $\alpha = t$ as time, we obtain a continuous-time jump process $(\mathbf{X}_t)_{t\geq 0}$, where we have "evolution by accretion" and "i.i.d. increments".

The process is constructed as follows: $(\mathbf{X}_t)_{t\geq 0}$ is constant almost everywhere except at the jump times, which are independent of the jumps; the jump times constitute a homogeneous Poisson process on $\mathbb{R}_0$ with rate D; the jumps are i.i.d. point processes with density $f_0(\mathbf{x}|\text{not } \emptyset)$; and $\mathbf{X}_t$ is the superposition of the jumps happening before or at time t. Note that $\mathbf{X}_0 = \emptyset$.

By Corollary 2, $\mathbf{X}_t$ is a permanent process with density f_t. The jump process is clearly Markovian and increasing ($\mathbf{X}_s \subseteq \mathbf{X}_t$ if $0 \leq s < t$). Hence, for each $0 \leq s \leq s + t$, the increments $\mathbf{X}_s$ and $\mathbf{X}_{s+t} \setminus \mathbf{X}_s$ are independent with densities f_s and f_t, respectively.

4.2 Second temporal extension

In this section we assume that 2α is a positive integer and the conditions of Section 2.2 are satisfied.

Consider a spatio-temporal Cox process for which the conditional intensity function at (x, t) is $\Lambda(x)$, constant in time, where $\Lambda(x)$ is given by (2). Let $t > 0$ be fixed, and let $\mathbf{X} \subset S$ be the set of points occurring in $[0, t]$. That is to say, $\mathbf{X}$ records the position of each point, but not the time of occurrence or the sequential order. Given $\mathbf{\Lambda}$, the process is Poisson with intensity function $t\Lambda(x)$ for $x \in S$. By Theorem 2, the density of $\mathbf{X}$ is given by (11) with λ_r replaced

by $t\lambda_r$. Thus $\tilde{\lambda}_r = t\lambda_r/(1+t\lambda_r)$, and the corresponding covariance function is denoted by $\tilde{C}_t$. Given that $n(\mathbf{X}) = n$, the conditional density of the points in S^n is proportional to $\mathrm{per}_\alpha[\tilde{C}_t](x_1,\ldots,x_n)$.

For inverse sampling, the number of points is fixed, and the process is observed until the time T_n at which $n \geq 1$ points have occurred. What then is the joint density of T_n and the n points? Let $\Gamma = \int_S \Lambda(x)\,\mathrm{d}x$. Given $\mathbf{\Lambda}$, the points are i.i.d. in S with density $\Lambda(x)/\Gamma$, and T_n has the gamma distribution with shape parameter n and mean n/Γ, independent of the n points. The conditional joint density at $(x_1,\ldots,x_n,t)$ is thus

$$\frac{\Lambda(x_1)\cdots\Lambda(x_n)}{\Gamma^n} \times \frac{t^{n-1}\Gamma^n \exp(-t\Gamma)}{(n-1)!}.$$

By the proof of Theorem 2, the unconditional joint density is

$$\frac{t^{-1}\,\mathrm{per}_\alpha[\tilde{C}_t](x_1,\ldots,x_n)}{(n-1)!\prod_0^\infty(1+t\lambda_r)^\alpha},$$

and the marginal density on S^n of the points is

$$f_n(x_1,\ldots,x_n) = \int_0^\infty \frac{t^{-1}\,\mathrm{per}_\alpha[\tilde{C}_t](x_1,\ldots,x_n)}{(n-1)!\prod_0^\infty(1+t\lambda_r)^\alpha}\,\mathrm{d}t. \tag{24}$$

As shown in Section 5.1, unless C is proportional to a projection, this is different from the conditional density obtained in the preceding paragraph.

The eigenvalues $\tilde{\lambda}_r = t\lambda_r/(1+t\lambda_r)$ of $\tilde{C}_t$ are strictly less than one, but increasing in t with limit one if $\lambda_r > 0$ and zero otherwise as $t \to \infty$. Thus, if C has finite rank, the limit $\lim_{t\to\infty}\tilde{C}_t$ is the orthogonal projection having the same range as C. Moreover, if for example $\lambda_r = \exp(-r)$ and $\delta > 0$, then the eigenvalues of $\tilde{C}_t$ are near one for $r < \log t - \delta\log\log t$ and near zero for $r > \log t + \delta\log\log t$. We interpret this result as saying that $\tilde{C}_t$ is approximately a projection of rank $\log t$ when t is large. These special permanent processes are studied in the next section.

5 The special permanent process

Let Q be a projection of rank m:

$$\int_S Q(x_1,x)Q(x,x_2)\,\mathrm{d}x = Q(x_1,x_2) \tag{25}$$

and $\int Q(x,x)\,dx = m$. This means that Q has m unit eigenvalues and the others are zero. It is assumed throughout this section that $\kappa > 0$ is a parameter and the covariance function $C = \kappa Q$ is a positive multiple of the projection; equivalently, $\tilde{C} = (\kappa/(1+\kappa))Q$. The associated point process $\mathbf{X}_\alpha$ is called special on account of its striking and unusual properties.

5.1 The density function revisited

Corollary 3 Suppose $\alpha > 0$. For any finite point configuration $\mathbf{x} \subset S$, the special permanent process $\mathbf{X}$ has density

$$f(\mathbf{x}) = \mathrm{e}^{|S|}(1+\kappa)^{-n(\mathbf{x})-\alpha m} \operatorname{per}_\alpha[C](\mathbf{x}) \tag{26}$$

and the number N of points in $\mathbf{X}$ follows a negative binomial distribution,

$$p_n = \frac{\Gamma(n+m\alpha)}{\Gamma(m\alpha)\, n!} \left(\frac{\kappa}{1+\kappa}\right)^n \left(\frac{1}{1+\kappa}\right)^{m\alpha}, \quad n = 0, 1, \ldots. \tag{27}$$

Furthermore, conditional on $N = n$, the joint density of the n points in $\mathbf{X}$ is

$$f_n(x_1, \ldots, x_n) = \operatorname{per}_\alpha[Q](x_1, \ldots, x_n)\Gamma(m\alpha)/\Gamma(n+m\alpha) \tag{28}$$

and

$$\int_S f_{n+1}(x_1, \ldots, x_n, x)\, \mathrm{d}x = f_n(x_1, \ldots, x_n). \tag{29}$$

Proof: We have $\operatorname{per}_\alpha[\tilde{C}](\mathbf{x}) = \operatorname{per}_\alpha[C](\mathbf{x})/(1+\kappa)^{n(\mathbf{x})}$. Since $\tilde{\lambda}_r = \kappa/(1+\kappa)$ for m eigenvalues and $\tilde{\lambda}_r = 0$ otherwise, $\prod_0^\infty (1+\lambda_r) = (1+\kappa)^m$. Hence (26) follows immediately from Theorem 3. By (21) N has cumulant generating function $-m\alpha \log(1+\kappa(1-\mathrm{e}^t))$ from which (27) follows. Furthermore, (28) follows from (26) and (27) and the usual relation between f and $p_n f_n$:

$$f(\mathbf{x}) = p_n f_n(x_1, \ldots, x_n) \exp(|S|)/n!.$$

Finally, (29) follows straightforwardly from (25) and (28).

Equation (29) is Kolmogorov's consistency condition for a stochastic process with marginal densities f_n. In other words, to each projection Q there corresponds an infinitely exchangeable process taking values in S, for which the n-dimensional joint density is f_n. Further, (26) implies that $(n(\mathbf{x}), \rho^{(n(\mathbf{x}))}(\mathbf{x}))$ is a minimal sufficient statistic, and equation (28) states that f_n is proportional to $\rho^{(n)}$. In general, for other non-trivial Cox processes such as log-Gaussian or shot-noise processes, no simple relationship exists connecting product densities with the density of the process.

Consider again the space-time setting in Section 4.2, where now $C = \kappa Q$. Suppose that the point configuration $\mathbf{x} = \{x_1, \ldots, x_n\}$ has been observed by inverse sampling with fixed n, and that we wish to predict where the next point X_{n+1} is likely to occur. Since the density f_n in (24) reduces to that in (28), and since f_n is the marginal density of f_{n+1}, the conditional density of X_{n+1} at x is

$$\frac{f_{n+1}(x_1, \ldots, x_n, x)}{f_n(x_1, \ldots, x_n)} = \frac{\operatorname{per}_\alpha[Q](x_1, \ldots, x_n, x)}{(m\alpha+n)\operatorname{per}_\alpha[Q](x_1, \ldots, x_n)}. \tag{30}$$

This predictive density is in fact the Bayes estimate of the intensity function $\Lambda(x)/\Gamma$, i.e. the conditional expected value of the normalized intensity function at x given the observed point configuration $\mathbf{x}$.

5.2 Palm distributions

We shall use the following notation. For any non-empty and finite point configuration $\mathbf{x} \subset S$, let $\mathrm{P}^!(\cdot|\mathbf{x})$ denote the nth order reduced Palm distribution of the special permanent process at $\mathbf{x}$. Intuitively, this is the conditional distribution of $\mathbf{X} \setminus \mathbf{x}$ given that $\mathbf{x} \subseteq \mathbf{X}$; a formal definition is given in Appendix A (the Campbell-Mecke theorem (35)-(36)). Finally, $\mathbf{\Phi}$ denotes the unit rate Poisson on S and $p_l(\mathbf{x})$ is the probability of observing l points under $\mathrm{P}^!(\cdot|\mathbf{x})$.

Corollary 4 Under the special permanent process, for any $\alpha > 0$ and non-empty finite point configuration $\mathbf{x} \subset S$ with $\mathrm{per}_\alpha[C](\mathbf{x}) > 0$, the reduced Palm distribution at $\mathbf{x}$ is

$$\mathrm{P}^!(F|\mathbf{x}) = \mathrm{E}\left[\frac{1[\mathbf{\Phi} \in F]\exp(|S|)\,\mathrm{per}_\alpha[C](\mathbf{\Phi} \cup \mathbf{x})}{(1+\kappa)^{\alpha m + n(\mathbf{x}) + n(\Phi)}\,\mathrm{per}_\alpha[C](\mathbf{x})}\right] \tag{31}$$

where F is any event of point configurations. Particularly, if $\alpha = 1$ and $n(\mathbf{x}) = n$,

$$p_l(\mathbf{x}) = \frac{(l+m+n-1)!}{(m+n-1)!l!}\left(\frac{\kappa}{1+\kappa}\right)^l\left(\frac{1}{1+\kappa}\right)^{m+n}, \quad l = 0, 1, \ldots. \tag{32}$$

Proof: See Appendix A.

When $\alpha = 1$ we consider the "special boson process" and it can be shown that (31)-(32) remain true if $\mathbf{Z}_1 + i\mathbf{Z}_2$ is a complex Gaussian process. Note that (32) extends (27). In words, conditional on the event that $\mathbf{X}$ contains $\mathbf{x}$, the distribution of the number of points in $\mathbf{X} \setminus \mathbf{x}$ is negative binomial and depends only on $\mathbf{x}$ through the number of points in $\mathbf{x}$. The latter property is remarkable: In general, for any point process with density f with respect to the unit rate Poisson process on S,

$$\mathrm{P}^!(F|\mathbf{x}) = \mathrm{E}\left[1[\mathbf{\Phi} \in F]f(\mathbf{\Phi} \cup \mathbf{x})\right]/\rho^{(n)}(\mathbf{x})$$

provided $n(\mathbf{x}) = n$ and $\rho^{(n)}(\mathbf{x}) > 0$ (this follows easily by modifying the first part of the proof above). For a Poisson process, by (35)-(37), $\mathrm{P}^!(F|\mathbf{x})$ is the distribution of the process itself independent of the points in $\mathbf{x}$. For a mixed Poisson process driven by the random intensity function $\mathbf{\Lambda}(x) = Rq(x)$, where R is a positive random variable and q is a deterministic function,

$$f(\mathbf{x}) = \mathrm{E}\left[\exp\left(|S| - R\int_S q(x)\,\mathrm{d}x\right)R^n\prod_1^n q(x_j)\right]$$

and

$$\rho^{(n)}(\mathbf{x}) = \mathrm{E}\left[R^n\prod_1^n q(x_j)\right],$$

so the reduced Palm distribution

$$\mathrm{P}^!(F|\mathbf{x}) = \mathrm{E}\left[1[\mathbf{\Phi} \in F]\exp\left(|S| - R\int_S q(x)\,\mathrm{d}x\right)R^{n+n(\mathbf{\Phi})}\right]/\mathrm{E}\left[R^n\right]$$

depends only on $\mathbf{x}$ through $n(\mathbf{x})$. For the special boson process, $\mathrm{P}^!(F|\mathbf{x})$ may depend on the locations of the points in $\mathbf{x}$, but $p_l(\mathbf{x})$ depends only on $n(\mathbf{x})$. Apart from these three cases we are not aware of any other kind of Cox process where $p_k(\mathbf{x})$ depends only on $n(\mathbf{x})$. For instance, for a boson process where C is not proportional to a projection,

$$p_0(\mathbf{x}) = \mathrm{P}^!(\emptyset|\mathbf{x}) = \frac{\mathrm{e}^{-D}\,\mathrm{per}[\tilde{C}](\mathbf{x})}{\mathrm{per}[C](\mathbf{x})}$$

depends on the locations of the points in $\mathbf{x}$ (here $\mathrm{per}[C](\mathbf{x}) = \mathrm{per}_1[C](\mathbf{x})$ is the usual permanent).

6 The determinant process

An analogous theory in which the fermion process replaces the boson process follows similar lines, extending the work of Diaconis and Evans (2000) in a different direction. We sketch this below.

Suppose that C satisfies the conditions of Section 2.2, i.e. C is a covariance function with spectral representation (7) such that $\sum_0^\infty \lambda_r < \infty$. The fermion (or electron) process is a finite point process with density

$$\tilde{f}_1(\mathbf{x}) = \mathrm{e}^{|S|-D}\det[C](\mathbf{x})$$

with respect to the unit rate Poisson process on S, and its nth order product density $\tilde{\rho}^{(n)}$ is given by

$$\tilde{\rho}^{(n)}(\mathbf{x}) = \det[\tilde{C}](\mathbf{x})$$

(Benard & Macchi 1973, Macchi 1975, Daley & Vere-Jones 2003). Note that $\det[C](\mathbf{x})$ and $\det[\tilde{C}](\mathbf{x})$ can be negative if C is not positive semi-definite.

The determinant polynomial

$$\det_\alpha[C](x_1,\ldots,x_n) = \mathrm{per}_{-\alpha}[-C](x_1,\ldots,x_n),$$

with $\alpha^{\#\sigma}\,\mathrm{sign}(\sigma) = (-1)^n(-\alpha)^{\#\sigma}$ in place of $\alpha^{\#\sigma}$, also satisfies the semi-group convolution property (20). Consequently, for positive integer α the family of point processes with density

$$\tilde{f}_\alpha(\mathbf{x}) = \mathrm{e}^{|S|-\alpha D}\det_\alpha[C](\mathbf{x}) \tag{33}$$

is closed under independent superposition. A point process with density $\tilde{f}_\alpha$ is called a determinant process. A Poisson process is obtained in the uncorrelated case, i.e. when $C(x,x') = 0$ whenever $x \neq x'$.

Most of the results established for permanent processes have a dual form for determinant processes with C and $\tilde{C}$ interchanged. Here are some examples: Let M denote the number of points in the determinant process. Its cumulant generating function

$$\log \mathrm{E}\left[\mathrm{e}^{tM}\right] = \alpha\sum_{r=0}^{\infty}\log(1+\mathrm{e}^t\lambda_r) - \alpha D, \quad t \leq -\log\|C\|,$$

can be obtained directly from the density (33). Thus M is distributed as the sum of independent binomial random variables with index α and parameter $\lambda_r/(1+\lambda_r)$. Consequently, M is under-dispersed and

$$M \leq \alpha \times \mathrm{rank}(C), \quad \mathrm{E}M = \alpha \sum_{r=0}^{\infty} \frac{\lambda_r}{1+\lambda_r}, \quad \mathrm{Var}M = \alpha \sum_{r=0}^{\infty} \frac{\lambda_r}{(1+\lambda_r)^2}.$$

By the argument used in Section 3.2, the nth order product density is

$$\tilde{\rho}^{(n)}(\mathbf{x}) = \det_\alpha[\tilde{C}](\mathbf{x}). \tag{34}$$

In particular, the intensity function is $\alpha\tilde{C}(x,x)$ and the pair correlation function is $1-[\tilde{C}(x,x')^2/(\tilde{C}(x,x)\tilde{C}(x,x'))]/\alpha$ (provided that $\tilde{C}(x,x)>0$ and $\tilde{C}(x',x')>0$), which is at least $1-1/\alpha$. Thus, in contrast to the permanent process and unless the process is Poisson, the points of the determinant process tend to repel one another, and the degree of repulsion is a decreasing function of α. Further, the special determinant process in which $C=\kappa Q$ is proportional to a projection of rank m has conditional densities

$$\det_\alpha[Q](x_1,\ldots,x_n)\Gamma(m\alpha-n+1)/\Gamma(m\alpha+1)$$

for $n \leq m\alpha$ only, and these satisfy the Kolmogorov consistency condition up to this order. Furthermore, M is binomial with index $m\alpha$ and parameter $\kappa/(1+\kappa)$, making it clear that the special determinant process is defined for integer α only.

In general the determinant process cannot be extended to $\alpha \in (0,1)$: if we claim f_α to be a density, then the pair correlation function is as above, so e.g. continuity of C (or equivalently $\tilde{C}$) implies that $1-1/\alpha \geq 0$, and hence $\alpha \geq 1$. Simulation along similar lines as in Section 3.3 seems therefore not possible. However, the usual birth-death-move Metropolis-Hastings algorithm (Geyer & Møller 1994) may be computationally feasible, since the Papangelou conditional intensity is a ratio of weighted determinants. It is not clear whether any determinant process can be extended to non-integer $\alpha \geq 1$.

Acknowledgments

Comments by Alexander Barvinok, Horia Cornean, Arne Jensen, Mark Jerrum, Gareth Roberts, Alistar Sinclair and Eric Vigoda are acknowledged. JM thanks Department of Statistics, University of Chicago, for kind hospitality. Funded by the Danish Natural Science Research Council and NSF Grant No. DMS-0305009.

Appendix A

By definition, the nth order reduced Palm distribution satisfies the integral equation:

$$\mathrm{E}\left[\sum h(x_1,\ldots,x_n,\mathbf{X}\setminus\mathbf{x})\right] \tag{35}$$

$$= \int_S \cdots \int_S \left(\int h(x_1,\ldots,x_n,\mathbf{y})\,\mathrm{d}P^!(\mathbf{y}|\mathbf{x}) \right) \rho^{(n)}(\mathbf{x})\,\mathrm{d}x_1\cdots\mathrm{d}x_n \tag{36}$$

for all measurable non-negative functions h, where the sum in (35) is over all pairwise different points $x_1, \ldots, x_n$ in $\mathbf{X}$ and $\mathbf{x} = \{x_1, \ldots, x_n\}$. The similar integral equation for the unit rate Poisson process is the Slivnyak-Mecke theorem:

$$\mathrm{E}\left[\sum h(x_1, \ldots, x_n, \mathbf{\Phi} \setminus \mathbf{x})\right] = \int_S \cdots \int_S \mathrm{E}\left[h(x_1, \ldots, x_n, \mathbf{\Phi})\right] \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \quad (37)$$

where the sum is over pairwise different points $x_1, \ldots, x_n$ in $\mathbf{\Phi}$. See, for example, Møller and Waagepetersen (2003).

Now, (35) is equal to

$$\begin{aligned}
&\mathrm{E}\left[\sum h(x_1, \ldots, x_n, \mathbf{\Phi} \setminus \mathbf{x}) f(\mathbf{\Phi})\right] \\
&= \int_S \cdots \int_S \mathrm{E}\left[h(x_1, \ldots, x_n, \mathbf{\Phi}) f(\mathbf{\Phi} \cup \mathbf{x})\right] \, \mathrm{d}x_1 \cdots \mathrm{d}x_n \\
&= \int_S \cdots \int_S \mathrm{E}\left[\frac{h(x_1, \ldots, x_n, \mathbf{\Phi}) \exp(|S|) \operatorname{per}_\alpha[C](\mathbf{\Phi} \cup \mathbf{x})}{(1+\kappa)^{m+n+n(\mathbf{\Phi})}}\right] \, \mathrm{d}x_1 \cdots \mathrm{d}x_n
\end{aligned}$$

using (37) in the first equality and (26) in the second equality. Comparing this with (36), we obtain (31).

Suppose $\alpha = 1$ and $n(\mathbf{x}) = n$. By (5) and (31),

$$\begin{aligned}
p_l(\mathbf{x}) &= \mathrm{E}\left[\frac{1[n(\mathbf{\Phi}) = l] \exp(|S|) \operatorname{per}[C](\mathbf{\Phi} \cup \mathbf{x})}{\operatorname{per}[C](\mathbf{x})(1+\kappa)^{l+m+n}}\right] \\
&= \int_S \cdots \int_S \frac{\operatorname{per}[C](\mathbf{x} \cup \mathbf{y})}{l! \operatorname{per}[C](\mathbf{x})(1+\kappa)^{l+m+n}} \, \mathrm{d}y_1 \cdots \mathrm{d}y_l \\
&= \frac{\mathrm{E}\left[\Gamma^{2l} \prod_1^n \Lambda(x_j)\right]}{l! \operatorname{per}[C](\mathbf{x})(1+\kappa)^{l+m+n}}
\end{aligned}$$

using a notation as in Section 4.2 with $k = 2$, and where we have used the projection property (25) and the fact that $\alpha = 1$ to obtain the last equation. By (9) and Basu's theorem, Γ is $(\kappa/2)\chi^2(2m)$-distributed and independent of $\mathbf{\Lambda}/\Gamma$. Thus

$$\mathrm{E}\left[\Gamma^{2l} \prod_1^n \Lambda(x_j)\right] = \frac{\mathrm{E}\left[\Gamma^{2(l+n)}\right] \mathrm{E}\left[\prod_1^n \Gamma(x_j)\right]}{\mathrm{E}\left[\Gamma^{2n}\right]} = \kappa^l \frac{(l+m+n-1)! \operatorname{per}[C](\mathbf{x})}{(m+n-1)!}$$

whereby (32) follows.

References

Baddeley, A., Møller, J. & Waagepetersen, R. (2000), 'Non- and semi-parametric estimation of interaction in inhomogeneous point patterns', *Statistica Neerlandica* **54**, 329–350.

Barvinok, A. I. (1996), 'Two algorithmic results for the traveling salesman problem', *Mathematics of Operations Research* **21**, 65–84.

Benard, C. & Macchi, O. (1973), 'Detection and emission processes of quantum particles in a chaotic state', *Journal of Mathematical Physics* **14**, 155–167.

Brix, A. (1999), 'Generalized gamma measures and shot-noise Cox processes', *Advances in Applied Probability* **31**, 929–953.

Cox, D. R. & Isham, V. (1980), *Point Processes*, Chapman & Hall, London.

Daley, D. J. & Vere-Jones, D. (2003), *An Introduction to the Theory of Point Processes. Volume I: Elementary Theory and Methods*, second edn, Springer-Verlag, New York.

Diaconis, P. & Evans, S. (2000), 'Immanants and finite point processes', *Journal of Combinatorial Theory* **91**, 305–321.

Diggle, P. J. (2003), *Statistical Analysis of Spatial Point Patterns*, second edn, Arnold, London.

Geyer, C. J. & Møller, J. (1994), 'Simulation procedures and likelihood inference for spatial point processes', *Scandinavian Journal of Statistics* **21**, 359–373.

Grandell, J. (1976), *Doubly Stochastic Poisson Processes*, Springer Lecture Notes in Mathematics 529. Springer-Verlag, Berlin.

Jerrum, M. R., Sinclair, A. & Vigoda, E. (2004), 'A polynomial-time approximation algorithm for the permanent of a matrix with non-negative entries', *Journal of the Association for Computing Machinery* **51**, 671–697.

Lieshout, M. N. M. van (2000), *Markov Point Processes and Their Applications*, Imperial College Press, London.

Macchi, O. (1971), 'Distribution statistique des instants d'mission des photolectrons d'une lumire thermique', *C. R. Acad. Sci. Paris Ser. A* **272**, 437–440.

Macchi, O. (1975), 'The coincidence approach to stochastic point processes', *Advances in Applied Probability* **7**, 83–122.

Malyshev, V. (1980), 'Cluster expansions in lattice models of statistical physics and the quantum theory of fields', *Russian Mathematical Surveys* **35**, 1–62.

McCullagh, P. (1984), 'Tensor notation and cumulants of polynomials', *Biometrika* **71**, 461–476.

Minc, H. (1978), *Permanents*, Addison-Wesley, Reading, MA.

Møller, J. (2003), 'Shot noise Cox processes', *Advances in Applied Probability* **35**, 4–26.

Møller, J., Syversveen, A. R. & Waagepetersen, R. P. (1998), 'Log Gaussian Cox processes', *Scandinavian Journal of Statistics* **25**, 451–482.

Møller, J. & Waagepetersen, R. P. (2003), *Statistical Inference and Simulation for Spatial Point Processes*, Chapman and Hall/CRC, Boca Raton.

Reed, M. & Simon, B. (1980), *Methods of Modern Mathematical Physics. I. Functional Analysis*, Academic Press, New York.

Ripley, B. D. (1976), 'The second-order analysis of stationary point processes', *Journal of Applied Probability* **13**, 255–266.

Ripley, B. D. (1977), 'Modelling spatial patterns (with discussion)', *Journal of the Royal Statistical Society Series B* **39**, 172–212.

Stoyan, D., Kendall, W. S. & Mecke, J. (1995), *Stochastic Geometry and Its Applications*, second edn, Wiley, Chichester.

Stoyan, D. & Stoyan, H. (1995), *Fractals, Random Shapes and Point Fields*, Wiley, Chichester.

Valiant, L. G. (1979), 'The complexity of computing the permanent', *Theoretical Computer Science* **8**, 189–201.