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Basic methods for simulation of random variables: 2. Accept/rejection algorithm

Another simple and general simulation method is the *accept/rejection* (or just *rejection*) algorithm, also called *von Neumann sampling*.

Suppose we want to make a simulation Y from a (discrete or continuous — and uni- or multi-variate) density function f (in the discrete case $f(x)$ is the probability of the state x). Sometimes we only know f up to proportionality, i.e. $f = h/c$ where c is an unknown normalising constant and h is a so-called *unnormalised density* (many examples of this are given later in the course). In the discrete case, $c = \sum_{x\in\Omega} h(x)$, while in the continuous case, $c = \int h(x)dx$ (where the integral is one or higher dimensional). When c is in fact known, we can without loss of generality take $c = 1$ and $h = f$; we say then that h is a normalised density. We call f (or h) the *target density* (or unnormalised target density).

The requirement in rejection sampling is that there exists an unnormalised density g and a constant M such that for all states $x \in \Omega$,

$$h(x) \leq Mg(x).$$

We call g the (unnormalised) *instrumental density*. The idea is that g is chosen such that it is easy to sample from g and $h(x)/(Mg(x))$ can be considered as a probability with which we accept a draw from g:

Accept/rejection algorithm:

1. Generate $X \sim g$ and $U \sim$ unif(0,1).
2. Accept $Y = X$ if $U \leq h(X)/(Mg(X))$, else repeat 1.–2.

Exercise 1

1. Show that the accept/rejection algorithm produces a variable Y distributed according to f.
   Hint:
   $$P(Y \in F) = P\left(X \in F \middle| U \leq \frac{h(X)}{Mg(X)}\right).$$
2. Suppose that both h and g are normalised densities.
   (a) Show that $M \geq 1$ and the probability of acceptance in step 2. of the accept/rejection algorithm is given by $p = 1/M$.
   (b) Let N denote the number of trials before acceptance in step 2. Show that
   $$P(N = n) = p(1-p)^n, \qquad n = 0, 1, \ldots$$
   (this distribution is called the *geometric distribution*).
   (c) Show that the mean number of times we perform steps 1.–2. is given by
   $$E(N+1) = M.$$
   Hint: *Recall that for any number $q \in (-1, 1)$ we have that*
   $$\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$$
   and so
   $$\sum_{n=1}^{\infty} \frac{d}{dq} q^n = \frac{d}{dq} \sum_{n=1}^{\infty} q^n = \frac{d}{dq}\left(\frac{1}{1-q} - 1\right) = \frac{1}{(1-q)^2}.$$
3. Discuss what would be good and bad choices of instrumental densities.

Exercise 2

Suppose $h(x) = \exp(-x^2/2)$ is our unnormalised target density (you probably recognise this density as that of a standard normal distribution $N(0,1)$). Consider the unnormalised instrumental density

$$g(x) = \exp(-|x|), \qquad -\infty < x < \infty.$$

1. Show that $h(x)/g(x) \leq \sqrt{\mathrm{e}}$ and $E(N+1) = \sqrt{2\mathrm{e}/\pi} = 1.315$ (see Exercise 1).

2. How would you simulate from the instrumental density?
   Hint: *Show that we can take $X = WZ$, where Z is exponentially distributed with parameter 1 (i.e. $P(Z > z) = \exp(-z)$ for $z \geq 0$), and W is a uniform random variable on $\{-1, 1\}$ (i.e. $P(W = -1) = P(W = 1) = 1/2$).*

3. Implement the accept/rejection algorithm in R so that simulations from $N(0, 1)$ of length $n = 1000$ are produced and $E(N + 1)$ is estimated by its empirical mean.

Exercise 3

The beta distribution with parameters $\alpha > 0$ and $\beta > 0$ has a continuous density given by

$$f(x) = \frac{1}{B(\alpha, \beta)} x^{\alpha-1}(1-x)^{\beta-1}, \qquad 0 < x < 1,$$

where $B(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha + \beta)$ is the so-called complete beta function and $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1} \exp(-y) dy$ is the so-called gamma function.

1. Make plots in R of the target density f when $(\alpha, \beta) = (0.5, 0.5), (1, 0.5), (0.5, 1), (1, 1), (1, 2), (2, 1), (2, 2), (5, 10)$.
   Hint: `gamma` *is the gamma function in R.*

2. In this and the following questions we let $\alpha > 1$ and $\beta > 1$. Show that $f(x)$ is increasing for $x \leq (\alpha - 1)/(\alpha + \beta - 2)$ and decreasing for $x \geq (\alpha - 1)/(\alpha + \beta - 2)$.

3. Discuss how you would chooce an instrumental density.

4. Find $E(N + 1)$ (see Exercise 1) when $g(x) = 1$ for $0 \leq x \leq 1$ and $(\alpha, \beta) = (2, 2), (5, 10)$.
   Hint: $\Gamma(\alpha) = (\alpha - 1)!$ *when $\alpha \geq 1$ is an integer.*

5. Implement the accept/rejection algorithm in R. Compare with results produced by the R-function `rbeta`.