Search for directed strongly regular graphs

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Abstract

In this paper we use standard computer-search techniques to investigate the existence of two types of structured digraphs.

One type is the directed strongly regular graphs introduced by A. Duval. They have the property that the number of directed paths of length 2 from vertex x to vertex y depends only on whether there is an edge from x to y. We prove existence in the last open case in Duval's list of parametersets for directed strongly regular graphs with at most 20 vertices.

The second type of "strongly regular digraphs" are the normally regular digraphs introduced by the author. Such digraphs have the property that the number of common out-neighbours of two vertices only depends on whether they are adjacent. In this case we construct some digraphs and find that for 10 parametersets there exist no graph.

1 Normally Regular Digraphs.

A normally regular digraph (or NRD) is a digraph for which there exist numbers k, λ and μ, so that every vertex has out-degree k, a pair of vertices x and y have exactly λ common out-neighbours if x and y are adjacent and μ common out-neighbours if they are non-adjacent. It is assumed that the digraph is in fact oriented, i.e. there is at most one edge between a pair of vertices.

This definition can be restated in terms of the $\{0,1\}$-adjacency matrix, A, by the following equation:

$$AA^T = kI + \lambda(A + A^T) + \mu(J - I - A - A^T) \quad (1)$$

with the additional condition that $A + A^T$ is also a $\{0,1\}$-matrix.

It is shown in [7] that the adjacency matrix of a normally regular digraph is normal, i.e., the properties of out-degree and out-neighbours holds for in-degree and in-neighbours as well.

The number of vertices of a normally regular digraph is denoted by v. An easy counting argument shows that

$$\mu v = (k + \mu - \lambda)^2 - k + \mu - (\mu - \lambda)^2. \tag{2}$$

Since we do not allow 2-cycles, we also have

$$v \geq 2k + 1 \quad \text{and} \quad k \geq 2\lambda + 1. \tag{3}$$

1.1 The search algorithm.

Since it is diffucult to determine existence of normally regular digraphs without computer, we want to apply computer-search to normally regular digraphs with small parameters in order to find such graphs or to prove non-existence.

In the computer a digraph is represented by its adjacency matrix. Since we want to use an orderly search algorithm (see Read [12]) we choose one canonical representation of the digraph among all possible adjacency matrices of a digraph.

If the rows of an adjacency matrix are written in one line; row 1 followed by row 2, etc., then we may interpret this as a binary number. Among all the adjacency matrices of a graph, there is one matrix for which this number is largest. This matrix is said to be in maximal form.

We will always use the maximal form adjacency matrix of a normally regular digraph. This means we want to have the 1's as far left as possible in the matrix, with highest priority to the first row.

The first row of an adjacency matrix in maximal form of a normally regular digraphs has 0 on the first (diagonal) entry. The following k entries are 1, and the remaining entries are 0. The second row has 0 on the first entry (since $A + A^T$ is a $\{0, 1\}$ matrix) it has 0 the diagonal entry, and then it has 1 on the following λ entries (since vertex 1 and vertex 2 are adjacent), 0 on the next $k - \lambda - 1$ entries and then $k - \lambda$ entries with 1 and the remaining entries are 0.

Suppose we have filled in rows number $1, \ldots, r-1$. Then row number r must satisfy

- It has exactly k entries 1 and $v - k$ entries 0.
- Entry number r is 0.
- For $i < r$, entry i is 0 if entry r in row i is 1.
- If the matrix has 0 in entry (r, i) and in entry (i, r) then the dotproduct of row i and row r is μ, otherwise it is λ, for $i = 0, \ldots, r-1$.
- The first r rows of the matrix is in maximal form.

For each possible way to fill in row r we repeat this procedure with r replaced by $r+1$, etc., until either we find some r for which no row satisfies the conditions or else all v rows are completed and the matrix satisfies the condition for a normally regular digraph.

Even if the final result is that no normally regular digraph exist with a given parameterset, there may be a very large number of matrices with r rows that satisfies the conditions, for some $r < v$. It is therefore usefull to have some further conditions that must be satisfied in order to get a smaller number of matrices with r rows that satisfies all conditions. The only such condition that I know is the following:

- The dotproduct of any two columns is at most the maximum of μ and λ. If it is already known that the two columns correspond to vertices that are adjacent (non-adjacent) then the dotproduct is at most λ (μ).

1.2 The design case: $\mu = \lambda$ or $\mu = \lambda + 1$.

It follows from the above matrix-equation (equation 1) that in the case $\mu = \lambda$ the adjacency matrix A is also the incidence matrix of a symmetric $2-(v, k, \mu)$ design, and in the case $\mu = \lambda + 1$ the matrix $A + I$ is incidence matrix of a symmetric $2-(v, k+1, \mu)$ design.

Design	v	k	λ	μ	no. of NRD's
PG(2,2)	7	2	0	1	1
	7	3	1	1	1
Hadamard-design	11	4	1	2	0
	11	5	2	2	1
PG(2,3)	13	3	0	1	5
	13	4	1	1	4
Hadamard-design	15	6	2	3	0
	15	7	3	3	2
2-(16,6,2)	16	5	1	2	16
	16	6	2	2	4
Hadamard-design	19	8	3	4	0
	19	9	4	4	2
PG(2,4)	21	4	0	1	187
	21	5	1	1	> 1000
Hadamard-design	23	10	4	5	0
	23	11	5	5	37
2-(25,9,3)	25	8	2	3	≥ 1
	25	9	3	3	≥ 1
Hadamard-design	27	12	5	6	
	27	13	6	6	722

Table 1.

Table 1 contains the result of the computer-search for normally regular digraphs with $\mu = \lambda$ or $\mu = \lambda + 1$ in all the cases with $v \leq 30$ where designs with

$v > 2k$ exist (see Beth, Jungnickel and Lenz [1]). In this and the following tables empty entries in "the number of graphs column" means that a search has not been possible because of a very large number of cases. In other cases a partial search has found some graphs, and so we have a lower bound on the number of graphs.

In the case $v = 2k+1 = 4\lambda+3$ the digraph is a tournament, i.e., every pair of vertices is joined by an edge. Such tournaments are called doubly-regular tournaments or Hadamard-tournaments, since their existence is equivalent the existence of skew Hadamard matrices of order $v+1$, see Reid and Brown [13]. These tournaments are also strongly regular in the sense of Duval. Such tournaments with at most 27 vertices were also enumerated by Spence [14] (with the same result as here).

We note that the normally regular digraphs with $\mu = \lambda + 1$ related to Hadamard-design do not exist for $\mu \leq 5$ except in the case $\mu = 1$, where the Hadamard-design is PG(2,2).

1.3 The case $\mu \notin \{\lambda, \lambda+1, 0, k\}$.

It is proved in [7] and [9] that normally regular digraphs with $\mu = k$ or $\mu = 0$ are in one to one correspondence with Hadamard tournaments or certain sets of Hadamard tournaments, respectively.

We therefore consider normally regular digraphs with

$$\mu \notin \{0, k, \lambda, \lambda+1\}. \tag{4}$$

There are 56 parametersets with $v \leq 36$ satifying equation 2 and inequalities 3. 26 of these parametersets can be excluded by the following Bruck-Ryser type theorem, see [7].

Theorem 1 *Suppose there exist a normally regular digraph with parameters* (v, k, λ, μ).

- *If v is even then $\eta = k - \mu + (\mu - \lambda)^2$ is a square.*
- *If v is odd then equation $x^2 + (-1)^{\frac{v+1}{2}}\mu y^2 = \eta z^2$ has an integer solution $(x, y, z) \neq (0, 0, 0)$.*

Further 11 parameterset can be excluded by the following combinatorial theorem, see [7].

Theorem 2 *Suppose there exist a normally regular digraph with parameters* (v, k, λ, μ).

- *If $2\mu > k + \lambda$ then $v - 2k$ divides v.*
- *If $\lambda = 0$ then $k > 2\mu + \frac{1}{2} + \sqrt{2\mu + \frac{1}{4}}$, unless $\mu = k$ or $\mu = 1$.*

The remaining 19 parametersets are listed in Tabel 2, with the result of the computer-search.

v	k	λ	μ	no. of NRD's
19	6	1	3	1
21	8	3	2	1
23	8	2	4	0
25	8	3	1	0
27	8	1	4	0
27	10	3	5	≥ 1
28	9	2	4	
28	12	5	4	
29	7	2	1	4
29	12	5	3	
31	10	2	5	0
31	10	4	1	0
31	12	4	6	≥ 1
31	12	5	2	
35	10	1	5	0
35	12	3	6	
35	14	5	7	
36	7	0	2	2
36	10	3	2	

Table 2.

The case $(36, 7, 0, 2)$ is one the easiest in table 2. A complete search was done on a SUN Ultra 2 workstation in about 2 minutes. The case $(31, 10, 2, 5)$ took about 30 hours. The cases that have not been searched all seems to be must more difficult.

1.3.1 The graphs found in the search.

In the cases $(27, 10, 3, 5)$ and $(31, 12, 4, 6)$ Tabel 2 says that there is at least one graph in each case. These graphs were not found by this computer search. They were found in [7] as Cayley graphs of the groups $\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$ and $\mathbb{Z}_{31}$, respectively.

The graph with parameters $(19, 6, 1, 3)$ is the Cayley graph of $\mathbb{Z}_{19}$ generated by $\{1, 4, 6, 7, 9, 11\}$. It automorphism group has order 57 and is generated by the maps $i \mapsto i + 1$ and $i \mapsto 7i$.

The graph with parameters $(21, 8, 3, 2)$ has automorphism group of order 7. It can described as follows.

The vertexset is $\{a_i, b_i, c_i \mid i \in \mathbb{Z}_7\}$. It has edges

from a_i to $a_{i+1}, a_{i+2}, b_i, b_{i+1}, b_{i+3}, b_{i+5}, c_i, c_{i+1}$, for $i \in \mathbb{Z}_7$,

from b_i to $b_{i+1}, b_{i+2}, c_i, c_{i+3}, c_{i+4}, c_{i+5}, a_{i+1}, a_{i+3}$, for $i \in \mathbb{Z}_7$,

from c_i to $c_{i+3}, c_{i+6}, a_{i+1}, a_{i+3}, a_{i+4}, a_{i+6}, b_{i+5}, b_{i+6}$, for $i \in \mathbb{Z}_7$.

All the eigenvalues of this graph have multiplicity 1.

In the case $(29, 7, 2, 1)$ one of the four graphs (is a 4-class association scheme and thus) has four eigenvalues of multiplicity 7. It is the Cayley graph of $\mathbb{Z}_{29}$ generated by $\{x^4 \mid x \in \mathrm{GF}(29), x \neq 0\}$. The other three graphs each have automorphism group of order 7.

One of the graphs with parameters $(36, 7, 0, 2)$ was known previously, since it is a 3-class association scheme (i.e., it has the additional property that the number of directed paths of length 2 from vertex x to vertex y is 0 if $x \to y$, 4 if $x \leftarrow y$ and 1 if x and y are non-adjacent). It was found (in 1982–84) by Faradžev, Klin and Muzichuk, who also found its automorphism group, $\mathrm{PSU}(3,3)$ of order 6048, see Faradžev, Klin and Muzichuk [3], page 115.

Goldbach and Claasen [5] showed that it is unique as an association scheme.

The other normally regular digraph with parameters $(36, 7, 0, 2)$ has vertexset $\{x_{i,j}, y_{i,j} \mid i = 1, \dots, 6, j \in \mathbb{Z}_3\}$, and edges

$$x_{i,j} \to x_{i,j+1} \quad \text{and} \quad y_{i,j} \to y_{i,j+1} \quad \text{for } i = 1, \dots, 6,\ j \in \mathbb{Z}_3 \tag{5}$$

and

$$x_{i,j} \to y_{k,j+1+i*k} \quad \text{and} \quad y_{i,j} \to x_{k,j+1+i*k} \quad \text{for } i,\ k = 1, \dots, 6, j \in \mathbb{Z}_3 \tag{6}$$

where $i * j \in \mathbb{Z}_3$ is defined by the multiplication table

	1	2	3	4	5	6
1	0	0	0	0	0	0
2	0	0	1	1	2	2
3	0	1	0	2	1	2
4	0	1	2	0	2	1
5	0	2	1	2	0	1
6	0	2	2	1	1	0

It is easy to see that this graph has $k = 7$. The common out-neighbours of $x_{i,j}$ and $y_{i',j'}$ is contained in $\{x_{i,k}, y_{i',k} \mid k \in \mathbb{Z}_3\}$, since the graph is nearly bipartite. $x_{i,j}$ and $x_{i,j'}$, $j \neq j'$ cannot have a common out-neighbour $y_{a,b}$, since it is not possible that $b = j + 1 + i * a$ and $b = j' + 1 + i * a$. $x_{i,j}$ and $x_{i',j'}$, $i \neq i'$ have a common out-neighbour $y_{a,b}$ if $b = j + 1 + i * a = j' + 1 + i' * a$. But this equation has two solutions, since row i and row i' differ by $j - j'$ in exactly two columns in the multiplication table. Since the map $x_{i,j} \mapsto y_{i,j}$ and $y_{i,j} \mapsto x_{i,j}$ is clearly an automorphism, $y_{i,j}$ and $y_{i',j'}$ also have two (no) common out-neighbours if $i \neq i'$ $(i = i')$. It follows that $\lambda = 0$ and $\mu = 2$.

The automorphism group has order 2160. It is transtive on vertices. The edges in equation 5 and 6 forms two edgeorbits.

2 Directed Strongly Regular Graphs.

A directed strongly regular graph is a digraph for which there exist numbers n, k, μ, λ, t such that the digraph has n vertices, every vertex has out-degree and in-degree k, the number 2-cycles incident with a vertex is t, the number of directed paths of length 2 from vertex x to vertex y is λ if there is an edge directed from x to y, and it is μ otherwise.

The adjacency matrix A satisfies

$$A^2 = tI + \lambda A + \mu(J - I - A),$$

and

$$AJ = JA = kJ.$$

Directed strongly regular graphs were defined by Duval [2]. Earlier Hammersley [6] had considered the special case $\mu = \lambda = 1$.

2.1 The search algoritm.

We want to use an orderly search algoritm similar to the one used for normally regular digraphs. But the problem is that comparing two rows of the adjacency matrix does not give any information about the number of directed paths of length two between the corresponding vertices. We have to compare a row and a column!

This problem can be solved if we dont use the adjacency matrix A to represent the graph but the matrix $B = 2A + A^T$. Then row r of B contains information about all edges directed into and out from vertex r.

We always represent a directed strongly regular graph by its B-matrix in maximal form.

The first row of such a matrix has 0 in the first entry, 3 in the following t entries, then 2 in $k - t$ entries, 1 in $k - t$ entries and the remaining entries are 0.

To fill in row r, $1 < r \leq v$, assume that rows $1, \ldots, r-1$ are filled. The new row must satisfy

- It has t 3's, $k - t$ 2's, $k - t$ 1's and 0 in the remaining entries.
- It has 0 in the diagonal entry.
- If entry (i, r) is 3, 2, 1 or 0, respectively then entry i in row r should be 3, 1, 2, 0, respectively, for $i = 1, \ldots, r-1$.
- If entry (i, r) is 3 or 2 (1 or 0) then there are exactly λ (μ) columns with 3 or 2 in row i and 3 or 1 in row r, for $i = 1, \ldots, r-1$.
- If entry (r, i) is 3 or 2 (1 or 0) then there are exactly λ (μ) columns with 3 or 2 in row r and 3 or 1 in row i, for $i = 1, \ldots, r-1$.
- The first r rows is a matrix in maximal form.

2.2 Result of search.

Duval [2] made a table of all possible parametersets with $n \leq 20$ and $k < \frac{n}{2}$ satisfying the eigenvalue conditions and some other conditions.

In table 3 we list the same parametersets. The last column contains what was known about existence of graphs with the given parameterset, before our computer-search. This column also refers to the first paper that proved the (non) existence: D is Duval [2], H is Hammersley [6], KMMZ is Klin, Munemasa, Muzychuk and Zieschang [10], FKM is Fiedler, Klin and Muzychuk [4].

Column 6 of table 3 contains the number of non-isomorphic digraphs found in the search described above.

Note that the two previously known non-existence result were confirmed by this search. The case $(n, k, \mu, \lambda, t) = (14, 5, 2, 1, 4)$ was done in about 1 second on a SUN ultra 2 workstation. The case $(n, k, \mu, \lambda, t) = (16, 6, 3, 1, 3)$ took about 70 minutes.

n	k	μ	λ	t	no. of graphs	existence
6	2	1	0	1	1	yes, D
8	3	1	1	2	1	yes, H
10	4	2	1	2	16	yes, D
12	3	1	0	1	1	yes, D
12	4	2	0	2	1	yes, D
12	5	2	2	3	20	yes, D
14	5	2	1	4	0	no, KMMZ
14	6	3	2	3	16495	yes, D
15	4	1	1	2	5	yes, H
15	5	2	1	2	1292	
16	6	3	1	3	0	no, FKM
16	7	2	4	5	1	yes, D
16	7	3	3	4		yes, FKM
18	4	1	0	3	1	yes, D
18	5	1	2	3	2	yes, FKM
18	6	3	0	3	1	yes, D
18	7	3	2	5		yes, FKM
18	8	3	4	5		yes, D
18	8	4	3	4		yes, D
20	4	1	0	1	1	yes, D
20	7	2	3	4		yes, KMMZ
20	8	4	2	4		yes, D
20	9	4	4	5		yes, D

Table 3.

2.2.1 The case $(n, k, \mu, \lambda, t) = (15, 4, 1, 1, 2)$.

Hammersley [6] found one such graph, which we denote by G_1. This graph is a special case of a general constructions in [2] and [8]. The 2-regular subgraph of G_1 consisting of undirected edges is the union of a 10-cycle and a 5-cycle. Its automorphism group is the dihedral group of order 10.

Another graph G_2 has vertex set $\{v_0, \ldots, v_{14}\}$. The undirected edges form a 15-cycle $v_0 \longleftrightarrow v_1 \longleftrightarrow \ldots \longleftrightarrow v_{14} \longleftrightarrow v_0$. The directed edges are

$$v_{0+5h} \to v_{7+5h},\ v_{8+5h}$$

$$v_{1+5h} \to v_{3+5h},\ v_{11+5h}$$

$$v_{2+5h} \to v_{0+5h},\ v_{11+5h}$$

$$v_{3+5h} \to v_{5+5h},\ v_{9+5h}$$

$$v_{4+5h} \to v_{2+5h},\ v_{9+5h}$$

for $h = 0, 1, 2$. The automorphism group S_3 is generated by the maps $v_i \mapsto v_{i+5}$ and $v_i \mapsto v_{15-i}$.

A third directed strongly regular graph with these parameters is G_3 with vertex set $\{w_{-7}, \ldots, w_7\}$. The undirected edges form the 15-cycle $w_{-7} \longleftrightarrow w_{-6} \longleftrightarrow \ldots \longleftrightarrow w_7 \longleftrightarrow w_{-7}$. The directed edges are $w_x \to w_y,\ w_x \to w_z$ and (if $x \neq 0$) $w_{-x} \to w_{-y},\ w_{-x} \to w_{-z}$ where (x, y, z) is one of the triples $(0, 2, -2), (1, 7, -2)$, $(2, -5, -4)$, $(3, 5, -1)$, $(4, -1, 2), (5, 7, -3), (6, 4, -3), (7, -6, 0)$. The only non-trivial automorphism of G_3 is the map $w_i \mapsto w_{-i}$.

The remaining two graphs with parameters $(15, 4, 1, 1, 2)$ are G_2^* and G_3^*, where G_i^* is the graph obtained from G_i by reversing the direction of all edges. G_1^* is isomorphic to G_1.

2.2.2 The case $(n, k, \mu, \lambda, t) = (15, 5, 2, 1, 2)$.

When we started this search project the case (15,5,2,1,2) was the only case from Duval's table that was still open. We have settled the existence problem in this case by showing that there are exactly 1292 non-isomorphic graphs with these parameters. The matrix $B = 2A + A^T$ in maximal form of one of these graphs is shown below. This graph has no non-trial automorphism. All these graphs have small automorphism groups of order at most 5. The number of graphs with automorphism group of order 1, 2, 3, 4, 5 is 1174, 100, 10, 5, 3, respectively.

$$
\begin{bmatrix}
0 & 3 & 3 & 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
3 & 0 & 3 & 2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
3 & 3 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 2 \\
1 & 1 & 0 & 0 & 0 & 0 & 3 & 3 & 0 & 1 & 2 & 2 & 2 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 3 & 1 & 2 & 3 & 2 & 0 & 0 & 2 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 3 & 2 & 3 & 0 & 2 & 0 & 0 & 2 \\
2 & 0 & 2 & 3 & 3 & 2 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\
2 & 0 & 2 & 3 & 2 & 3 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\
2 & 0 & 0 & 0 & 1 & 1 & 2 & 0 & 0 & 0 & 2 & 1 & 3 & 3 & 0 \\
0 & 2 & 2 & 2 & 3 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 2 & 0 & 1 & 1 & 0 & 0 & 2 & 1 & 0 & 0 & 2 & 3 & 0 & 3 \\
0 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 2 & 1 & 0 & 0 & 3 & 3 \\
0 & 0 & 1 & 1 & 0 & 0 & 2 & 0 & 3 & 2 & 3 & 0 & 0 & 2 & 1 \\
0 & 0 & 1 & 0 & 1 & 0 & 0 & 2 & 3 & 2 & 0 & 3 & 1 & 0 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 2 & 2 & 0 & 0 & 3 & 3 & 2 & 1 & 0
\end{bmatrix}
$$

2.2.3 The case $(n, k, \mu, \lambda, t) = (2m^2, 2m-1, 1, m-1, m)$.

For every integer $m \geq 2$ we construct a directed strongly regular graph G_m with $2m^2$ vertices $\{x_{i,j}, y_{i,j} \mid i, j = 1, \ldots, m\}$. For each $i = 1, \ldots, m$ the sets $\{x_{i,1}, \ldots, x_{i,m}\}$ and $\{y_{i,1}, \ldots, y_{i,m}\}$ span complete graphs K_m. Furthermore the graph has edges $x_{i,j} \to y_{j,l}$ and $y_{i,j} \to x_{j,l}$ for every $i, j, l = 1, \ldots, m$. In particular the undirected edges between the complete subgraphs are $x_{i,j} \longleftrightarrow y_{j,i}$. (An algebraic construction of these graphs from generalized quadrangles is given by Klin, Pech and Zieschang [11]).

The map $\beta : x_{i,j} \mapsto y_{i,j},\ y_{i,j} \mapsto x_{i,j}$ is an automorphism and for any two permutations p and q of $\{1, \ldots, m\}$ the map $\alpha_{p,q} : x_{i,j} \mapsto x_{p(i),q(j)},\ y_{i,j} \mapsto y_{q(i),p(j)}$ is an automorphism. It follows that G_m is vertex transitive. For $m \geq 3$ is it easy to see that the automorphism group of G_m is imprimitive with three systems of blocks

- $X = \{x_{i,j} \mid i, j = 1, \ldots, m\},\ Y = \{y_{i,j} \mid i, j = 1, \ldots, m\}$.
- $X_i = \{x_{i,1}, \ldots, x_{i,m}\},\ Y_i = \{y_{i,1}, \ldots, y_{i,m}\},\ \ i = 1, \ldots, m$.
- $\{x_{i,j}, y_{j,i}\},\ \ i, j = 1, \ldots, m$.

(this is true even for $m = 2$).

Let τ be an arbitrary automorphism of G_m. If τ interchanges X and Y then we replace τ by $\tau\beta$. So we may assume that X and Y are fixed by τ as sets. There exist permutations p and q in S_m so that τ maps X_i to $X_{p(i)}$ and Y_i to $Y_{q(i)}$ for $i = 1, \ldots, m$. A vertex in X_i that dominates Y_j is mapped to a vertex in $X_{p(i)}$ that dominates $Y_{q(j)}$. Thus $\tau(x_{i,j}) = x_{p(i),q(j)}$. Similarly, $\tau(y_{i,j}) = y_{q(i),p(j)}$. Thus $\tau = \alpha_{p,q}$ and so $\{\alpha_{p,q}\beta^i \mid p, q \in S_m,\ i = 0, 1\}$ is the full group of automorphisms.

Every vertex in the graph has degree $k = 2m - 1$ and is incident with $t = m$ undirected edges.

Since G_m is vertex transitive we need only consider directed paths of length 2 starting at $x_{1,1}$. Let $x_{i,j}$ be a vertex so that we do not have $x_{1,1} \to x_{i,j}$, i.e., $i \neq 1$. A directed path $x_{i,j} \to z \to x_{i,j}$ must satisfy $z = y_{u,v}$ for some u, v. Since $x_{1,1} \to y_{u,v}$, $u = 1$. Since $y_{u,v} \to x_{i,j}$, $v = i$. Thus there is a unique path of length 2 from $x_{1,1}$ to $x_{i,j}$.

Let $y_{i,j}$ be a vertex so that $x_{1,1} \not\to y_{i,j}$, i.e., $i \neq 1$. Since $x_{1,1}$ does not have any out neighbour in $\{y_{i,1}, \ldots, y_{i,m}\}$, a vertex z so that $x_{1,1} \to z \to y_{i,j}$, must satisfy $y = x_{1,l}$ for some l. Since $x_{1,l} \to y_{i,j}$, $l = i$. Thus there is aunique path of length 2 from $x_{1,1}$ to $y_{i,j}$ and so $\mu = 1$.

The vertices z that satisfies $x_{1,1} \to z \to x_{1,j}$ are the vertices $z = x_{1,i}$, where $i \neq 1, j$, and $z = y_{1,1}$.

The vertices z that satisfies $x_{1,1} \to z \to y_{1,j}$ are the vertices $z = y_{1,i}$, where $i \neq j$. Thus $\lambda = m - 1$.

For $m = 2$, G_2 is the unique directed strongly regular graph with parameters $(8, 3, 1, 1, 2)$.

For any vertex $z \in G_m$ the set $\{v \in G_m \mid z \to v \not\to z\}$ spans a complete graphs with $m-1$ vertices, and the set $\{v \in G_m \mid v \to z \not\to v\}$ is an independent set. It follows that for $m > 2$ the graph G_m^* obtained from G_m by reversing the direction of all edges is not isomorphic to G_m.

For $m = 3$, G_3 and G_3^* are the only directed strongly regular graphs with parameters $(18, 5, 1, 2, 3)$.

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