Exercises for
Mathematics for Computer Graphics

On the following pages you will find a number of exercises to be used in the course "Mathematics for Computer Graphics" at MED5 (Medialogy) and SP1 (Computer Science), Aalborg University 2008. The course uses the very nice book

[VB] "Essential Mathematics for Games," Second Edition, James M. van Verth and Lars M. Bishop, Morgan Kaufmann

Although the CD-ROM of the book contains some exercises below you will find supplementary ones.

The present file will be updated throughout the course Mathematics for Computer Graphics. So stay updated.

1 Exercises for Chapter 2 of [VB]

Exercise 1 *Consider the vectors* $\vec{v} = (3, 1)$ *and* $\vec{w} = (0, 2)$. *Calculate* $\vec{v} - \vec{w}$ *and* $\vec{v} + \vec{w}$. *For these vectors make a drawing like Figure 2.3 at page 38 of [VB]. Check that the vectors* $\vec{v} - \vec{w}$ *and* $\vec{v} + \vec{w}$ *in the drawing corresponds to the vectors you calculated above.*

Exercise 2 *Consider the vectors* $\vec{u} = (1, 1, 0)$, $\vec{v} = (0, 2, 3)$ *and* $\vec{w} = (2, 4, 3)$.

1. *Express* $\vec{u}$ *as a linear combination of* $\vec{v}$ *and* $\vec{w}$.
2. *Express* $\vec{v}$ *as a linear combination of* $\vec{u}$ *and* $\vec{w}$
3. *Express* $\vec{w}$ *as a linear combination of* $\vec{u}$ *and* $\vec{v}$
4. *Express* $\vec{0}$ *as a non-trivial linear combination of* $\vec{u}$, $\vec{v}$ *and* $\vec{w}$.
5. *Are* $\vec{u}, \vec{v}, \vec{w}$ *linearly dependent or are they linearly independent?*
6. *Which geometric object does the set of vectors* $\{\vec{u}, \vec{v}, \vec{w}\}$ *span?*
7. *Find a basis for above mentioned span*
8. *Are some two of the vectors* $\vec{u}, \vec{v}, \vec{w}$ *parallel?*

Exercise 3

1. *Are the vectors* $\vec{u} = (1, 0, 1, 1)$, $\vec{v} = (0, 0, 1, 0)$ *and* $\vec{w} = (1, 0, 0, 1)$ *linearly dependent?*

2. *Are the vectors* $\vec{u} = (1, 0, 1, 1)$, $\vec{v} = (2, 4, 7, 12)$ *linearly dependent?*

Exercise 4

1. *Find the* l_1*-norm of* $(-1, 2, 4)$ *(the* l_1*-norm is also known as the Manhattan norm).*

2. *Find the Euclidean norm of* $(-1, 2, 4)$.

Exercise 5 *In this exercise we consider the* l_1*-norm (also known as the Manhattan norm). Let* $\vec{v} = (v_1, v_2, v_3)$ *and* $\vec{w} = (w_1, w_2, w_3)$. *Show that*

$$|| \vec{v} + \vec{w} ||_{l_1} \leq || \vec{v} ||_{l_1} + || \vec{w} ||_{l_1}$$

holds. You will need the well-known result that

$$| a + b | \leq | a | + | b |$$

holds for any real numbers a, b.

Exercise 6 *Normalize the vector* $\vec{v} = (1, 2, 1, 3, 1, 3)$.

Exercise 7 *This exercise requires a calculator. Find the angle between* $(1, 2, 3)$ *and* $(0, 1, 4)$.

Exercise 8

1. *Are the vectors* $\vec{u} = (1, 2, 3), \vec{v} = (3, 0, -1)$ *orthogonal (also called perpendicular)?*

2. *Are the vectors* $\vec{u} = (2, 4, 0), \vec{v} = (1, -3, 7)$ *orthogonal (also called perpendicular)?*

3. *Is the angle between* $\vec{u} = (2, 4, 0)$ *and* $\vec{v} = (3, 0, -1)$ *greater or smaller than* 90^o*? (you are not supposed to actually calculate the angle)*

Exercise 9 *In this exercise we consider a game where an agent wants to detect an enemy. The agent is in position* $A = (1, 0, 0)$ *and the enemy is in position* $E = (2, -2, 0)$. *Hence, the vector pointing from the agent to the enemy is* $\vec{t} = E - A = (1, -2, 0)$. *The agent is looking in the direction* $\vec{v} = (1, 1, 1)$. *Can the agent see the enemy? (The calculation you are going to do is the same as a computer would have to do when running a game).*

Exercise 10 *Let* $\vec{u} = (1, 2)$. *Find a vector* $\vec{v}$ *of the same length such that* $\vec{u}$ *and* $\vec{v}$ *are orthogonal.*

Exercise 11 *Let* $\vec{u} = (a, b)$. *Find a vector* $\vec{v}$ *of the same length such that* $\vec{u}$ *and* $\vec{v}$ *are orthogonal. How many possible choices are there for* $\vec{v}$*?*

Exercise 12 *Let* $\vec{u} = (1, 2, -1)$*. Find a vector* $\vec{v}$ *of the same length such that* $\vec{u}$ *and* $\vec{v}$ *are orthogonal. How many possible choices are there for* $\vec{v}$*?*

Exercise 13 *Let* $\vec{v} = (0, 1, 2)$ *and* $\vec{w} = (3, 0, 4)$*.*

1. *Find the projection of* $\vec{v}$ *onto* $\vec{w}$
2. *Find the part of* $\vec{v}$ *that is perpendicular to* $\vec{w}$
3. *Calculate the sum* $\text{proj}_{\vec{w}}\vec{v} + \text{perp}_{\vec{w}}\vec{v}$*. Explain the result you get.*

Exercise 14 *Orthogonalize*

$$\{\vec{v}_0 = (1, 0, 0), \vec{v}_1 = (1, 1, 0), \vec{v}_2 = (1, 1, 1)\}$$

by using Gram-Schmidt orthogonalization. Then normalize the resulting basis.

Exercise 15 *Orthogonalize*

$$\{\vec{v}_0 = (1, 0, 1), \vec{v}_1 = (1, 1, 1), \vec{v}_2 = (0, 1, 1)\}$$

by using Gram-Schmidt orthogonalization. Then normalize the resulting basis.

Exercise 16 *In this exercise we change the enumeration of the vectors in Exercise 15 and investigate what is the implication. Orthogonalize*

$$\{\vec{v}_0 = (1, 1, 1), \vec{v}_1 = (0, 1, 1), \vec{v}_2 = (1, 0, 1)\}$$

by using Gram-Schmidt orthogonalization. Then normalize the resulting basis.

Exercise 17 *Consider the dot product on* $\mathbf{R}^3$*. Show that the conditions 1.–5. on page 48 are satisfied.*

Exercise 18 *Define the dot product on* $\mathbf{R}^4$ *in an obvious way. Show that the conditions 1.–5. on page 48 are satisfied.*

Exercise 19 *In this exercise we want to extend*

$$\{\vec{u} = (1, 2, 3), \vec{v} = (-3, 1, 1)\}$$

to a basis for $\mathbb{R}^3$ *by adding a third vector* $\vec{w}$*.*

1. *Perform this task such that the resulting basis is right handed*
2. *Perform this task such that the resulting basis is left handed*

Exercise 20 *Let* $\vec{u} = (-2, 3, 1)$*,* $\vec{v} = (0, 4, 1)$*. Find the area of the parallelogram bordered by* $\vec{u}$ *and* $\vec{v}$*.*

Exercise 21 *Using the cross product check if* $\vec{u} = (2, 6, 12)$ *and* $\vec{v} = (1, 3, 6)$ *are parallel.*

Exercise 22 *Let the corners of a triangle be* $P = (0,0,0)$*,* $Q = (1,1,1)$ *and* $R = (0,2,0)$*. Find a normal* $\vec{n}$ *for the triangle.*

Exercise 23 *Use the triple product method from Section 2.2.9 to derive from*

$$\{\vec{u} = (1,0,1), \vec{v} = (0,1,1), \vec{w} = (1,1,1)\}$$

an orthogonal basis. If $\vec{u}$ *corresponds to the view direction and* $\vec{v}$ *corresponds to upwards, which of the new vectors you find should be used as approximation for "upwards"?*

Exercise 24 *Sketch the parallelepiped (box) having as sides the vectors* $\vec{u} = (1,0,2)$*,* $\vec{v} = (0,1,2)$ *and* $\vec{w} = (1,1,1)$*. Find its volume.*

Exercise 25 *Use the scalar triple product to decide if the basis*

$$\{(1,0,0), (0,1,-1), (0,1,1)\}$$

is right handed or left handed. Interchange the order of the vectors to get a basis behaving opposite to this.

Exercise 26 *In our reference frame upwards is defined by* $\vec{u} = (0,0,1)$*. A tank moves with velocity* $\vec{v} = (1,1,0)$*. We want it to move instead in the direction* $\vec{d} = (3,2,0)$*. Should we turn left or should we turn right?*

Exercise 27 *Determine the distance between* $P_1 = (-2,3,0)$ *and* $P_0 = (4,2,3)$*.*

Exercise 28 *As exercise 27 but now use instead of the Euclidean distance the Manhattan distance.*

Exercise 29 *Consider the triangle with corners* $P = (1,0,0)$*,* $Q = (1,1,1)$*,* $R = (0,2,0)$*. Find the centroid.*

Exercise 30

1. *Is a ball a convex set?*
2. *Is a banana a convex set?*

Exercise 31 *Let* $P_0 = (1,0,0)$*,* $P_1 = (1,1,1)$*,* $P_2 = (0,2,0)$*.*

1. *Show that* P_0, P_1, P_2 *is a simplex.*
2. *Find the barycentric coordinates of* P_0
3. *Find the barycentric coordinates of* P_1
4. *Which point has barycentric coordinates* $(-1,1,1)$*?*
5. *Consider the convex hull of* P_0, P_1, P_2*. What kind of geometric object do we have?*

6. *Find the centroid of the convex hull.*

Exercise 32 *Consider the triangle having corners* $P_0 = (1, 0, 0)$, $P_1 = (0, 2, 0)$, $P_2 = (-1, -1, 0)$. *The triangle contains the point* $P = (0, 0, 0)$. *Using the method from [VB] page 85 find the barycentric coordinates for* P.

Exercise 33 *Find the polar coordinates for the point* $(3, -4)$

Exercise 34 *Given the polar coordinates* $r = 2$ *and* $\theta = \frac{3\pi}{4}$ *find the corresponding Cartesian coordinates.*

Exercise 35 *Given the point* $(1, 1, 1)$ *find the corresponding spherical coordinates.*

Exercise 36 *Given the spherical coordinates* $\rho = 2$, $\phi = \frac{\pi}{4}$ *and* $\theta = \frac{\pi}{4}$ *find the corresponding Cartesian coordinates.*

Exercise 37 *Find at least three different parametric equations for the line through* $(1, 1, 1)$ *and* $(-2, 4, 16)$.

Exercise 38 *Given the line* $L(t) = (1, 1) + t(1, -1)$, $t \in \mathbf{R}$ *and the point* $Q = (4, 4)$ *find the distance between them.*

Exercise 39 *Show that the points* $(1, 0, 1)$, $(2, 1, -7)$, $(2, 3, 4)$ *are not collinear.*

Exercise 40 *A plane contains the point* $P_0 = (-2, 1, -1)$ *and has normal vector* $\vec{n} = (1, 2, 3)$. *Find a generalized plane equation for the plane.*

Exercise 41 *The points* $P_1 = (0, 0, 0)$, $P_2 = (1, 1, 1)$, $P_3 = (0, 1, 0)$ *are not collinear. Hence, they define a plane.*

1. *Find a generalized plane equation for the plane*
2. *Find a parametric equation for the plane*

Exercise 42 *Are the points* $P_1 = (0, 0, 0)$, $P_2 = (1, 1, 1)$, $P_3 = (0, 1, 0)$, $P_4 = (1, 0, 0)$ *coplanar? Make sure to explain why or why not.*

Exercise 43 *Consider the triangle having corners* $P_0 = (1, 0, 0)$, $P_1 = (0, 2, 0)$, $P_2 = (-1, -1, 0)$. *The point* $P = (0, 0, 0)$ *is contained in the plane defined by* P_0, P_1, P_2. *Using the method from [VB] pages 83-84 show that the point* $P = (0, 0, 0)$ *is contained in the triangle.*

2 Partial solutions to exercises for Chapter 2 of [VB]

Solution to Exercise 2:

1. $\vec{u} = -\frac{1}{2}\vec{v} + \frac{1}{2}\vec{w}$
2. $\vec{v} = -2\vec{u} + \vec{w}$
3. $\vec{w} = 2\vec{u} + \vec{v}$
4. $\vec{0} = 2\vec{u} + \vec{v} - \vec{w}$
5. Linearly dependent
6. A plane
7. Any two of the vectors $\vec{u}, \vec{v}, \vec{w}$ will do
8. No

Solution to Exercise 3:

1. Yes
2. No

Solution to Exercise 4:

1. 7
2. $\sqrt{21}$

Solution to Exercise 6: $(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}, \frac{3}{5}, \frac{1}{5}, \frac{3}{5})$.

Solution to Exercise 7: $0,434$ rad or $24,84^o$.

Solution to Exercise 8:

1. Yes
2. No
3. Smaller

Solution to Exercise 9: No.

Solution to Exercise 10: $(2, -1)$ or $(-2, 1)$

Solution to Exercise 11: Assuming not both a and b are zero we have two possibilities $(b, -a)$ and $(-b, a)$.

Solution to Exercise 12: Infinitely many choices.

Solution to Exercise 13:

1. $(\frac{24}{25}, 0, \frac{32}{25})$

2. $(-\frac{24}{25}, 1, \frac{18}{25})$

Solution to Exercise 14:

$$\hat{w}_0 = \vec{w}_0 = (1, 0, 0)$$

$$\hat{w}_1 = \vec{w}_1 = (0, 1, 0)$$

$$\hat{w}_2 = \vec{w}_2 = (0, 0, 1)$$

Solution to Exercise 15:

$$\vec{w}_0 = (1, 0, 1)$$

$$\vec{w}_1 = (0, 1, 0)$$

$$\vec{w}_2 = (-\frac{1}{2}, 0, \frac{1}{2})$$

$$\hat{w}_0 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$$

$$\hat{w}_1 = (0, 1, 0)$$

$$\hat{w}_2 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$$

Solution to Exercise 16:

$$\vec{w}_0 = (1, 1, 1)$$

$$\vec{w}_1 = (-\frac{2}{3}, \frac{1}{3}, \frac{1}{3})$$

$$\vec{w}_2 = (0, -\frac{1}{2}, \frac{1}{2})$$

$$\hat{w}_0 = (\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$$

$$\hat{w}_1 = (-\frac{\sqrt{2}}{\sqrt{3}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}})$$

$$\hat{w}_2 = (0, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$$

Solution to Exercise 19:

1. $\vec{u} \times \vec{w} = (-1, -10, 7)$

2. $(1, 10, -7)$

Solution to Exercise 20: $\sqrt{69}$

Solution to Exercise 21: Start by checking if the cross product is close to $\vec{0}$. If this is the case then the area of the parallelogram bordered by $\vec{u}$ and $\vec{v}$ is small, meaning that $\vec{u}$ and $\vec{v}$ are either close to being parallel or are close to being antiparallel. Finally, we check if the angle between $\vec{u}$ and $\vec{v}$ is greater than 90^o.

Solution to Exercise 22: More possibilities. One is $\vec{n} = (-2, 0, 2)$.

Solution to Exercise 23: $\vec{w}_0 = \vec{u}, \vec{w}_1 = \vec{u} \times \vec{w}, \vec{w}_2 = \vec{u} \times (\vec{u} \times \vec{w})$.
$\vec{w}_1$ "corresponds to upwards".

Solution to Exercise 24: 3

Solution to Exercise 25:

$$\{(1, 0, 0), (0, 1, -1), (0, 1, 1)\}$$

is right handed.
To get left handed, interchange the last two vectors.

Solution to Exercise 26: $\vec{u} \cdot (\vec{v} \times \vec{d}) < 0$ and therefore we must turn right.

Solution to Exercise 29: $(\frac{2}{3}, 1, \frac{1}{3})$

Solution to Exercise 30:

1. Yes

2. No

Solution to Exercise 31:

1. The vectors $P_1 - P_0$ and $P_2 - P_0$ are linearly independent.

2. $(1, 0, 0)$

3. $(0, 1, 0)$

4. $(0, 3, 1)$

5. The triangle with corners P_0, P_1, P_2

6. See solution to exercise 29

Solution to Exercise 32:
The barycentric coordinates are $(a_0 = \frac{2}{5}, a_1 = \frac{1}{5}, a_2 = \frac{2}{5})$ corresponding to $(s = \frac{1}{5}, t = \frac{2}{5})$.

Solution to Exercise 33: $r = 5$, $\theta = -0.9272...$

Solution to Exercise 34: $(-\sqrt{2}, \sqrt{2})$

Solution to Exercise 35: $\rho = \sqrt{3}, \phi = 0.9553..., \theta = \frac{\Pi}{4}$

Solution to Exercise 36: $(1, 1, \sqrt{2})$

Solution to Exercise 41:
First find the normal

$$\vec{n} = (P_2 - P_1) \times (P_3 - P_1) = (-1, 0, 1).$$

Then find the plane equation as

$$(x - 0, y - 0, z - 0) \cdot (-1, 0, 1) = 0$$
$$\Updownarrow$$
$$-x + z = 0$$

One parametric equation is

$$P(s,t) = P_1 + s(P_2 - P_1) + t(P_3 - P_1)$$
$$\Updownarrow$$
$$P(s,t) = s(1,1,1) + t(0,1,0)$$
$$\Updownarrow$$
$$P(s,t) = (s, s+t, s)$$

3 Exercises for Chapter 3 of [VB]

Exercise 44 *Consider the map $\tau : \mathbf{R}^3 \to \mathbf{R}^3$ given by*

$$\tau\big((a,b,c)^T\big) = (a+c, b+c, 0)^T$$

for all choices of a, b, c.

1. *Is it a linear transformation?*
2. *Determine the null space?*
3. *Determine nullity(τ) and rank(τ).*

Exercise 45 *This is a continuation of Exercise 44. Find the matrix A such that*

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \tau\big([x,y,z]^T\big)$$

Exercise 46 *A linear transformation $\tau : \mathbf{R}^3 \to \mathbf{R}^2$ is given by*

$$\begin{aligned} \tau\big((1,0,0)^T\big) &= (1,1)^T \\ \tau\big((0,1,0)^T\big) &= (2,3)^T \\ \tau\big((0,0,1)^T\big) &= (0,1)^T \end{aligned}$$

1. *Find $\tau\big((2,-7,4)^T\big)$.*

2. *Determine nullity(τ) and rank(τ).*

3. *Find the matrix A such that*

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \tau\big([x,y,z]^T\big)$$

Exercise 47 *Let*

$$A = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 2 & 2 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 2 & 2 \end{bmatrix}$$

Find $A + B$, $A - B$ and $B - A$.

Exercise 48 *Let*

$$A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & 4 & 6 \end{bmatrix}, B = \begin{bmatrix} 0 & 2 \\ 1 & 3 \\ -2 & 4 \end{bmatrix}$$

Find $A^T + B$ and $A + B^T$.

Exercise 49 *Determine the product*

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 & 7 \\ 0 & -2 & 4 & 0 \\ 1 & 7 & 2 & 2 \\ -1 & 2 & -1 & 1 \end{bmatrix}$$

using a method similar to the one described on the top of page 95 in [VB]. You should set $A = B = C = I_2$. Then do the same for the product

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 3 & 3 \\ 0 & -2 & 4 \\ 1 & 7 & 2 \\ -1 & 2 & -1 \end{bmatrix}$$

Exercise 50

1. *Find* $\begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ -1 & 3 \end{bmatrix}$

2. *Find* $\begin{bmatrix} 7 & -1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

3. How are the results in 1. and 2. related?

Exercise 51 *Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 & 0 \\ 1 & 0 & 7 \end{bmatrix}$$

Find BA^T and AB^T.

Exercise 52 *Let*

$$A = \begin{bmatrix} 0 & 2 & 1 & 5 \\ 0 & 3 & 4 & 0 \\ 0 & 1 & 2 & -7 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 2 & 2 \\ 2 & 2 \end{bmatrix}$$

Find AB.

Exercise 53 *Let*

$$A = \begin{bmatrix} 2 & 3 \\ 0 & 7 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 \\ 4 & 2 \end{bmatrix}$$

Find $AB - BA$.

Exercise 54 *Using the method from page the top of page 95 determine AB, where*

$$A = \begin{bmatrix} 2 & 3 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

Exercise 55 *In this exercise we use the tensor product to project onto the vector $\hat{v} = (\frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2})^T$. Project the vectors $\vec{u}_1 = (1, 2, 0)^T$, $\vec{u}_2 = (\sqrt{2}, \sqrt{2}, \sqrt{2})^T$.*

Exercise 56 *Let $\vec{v} = (1, 2, -2)^T$. Find the matrix $\tilde{\vec{v}}$. Using $\tilde{\vec{v}}$ calculate $\vec{v} \times \vec{w}$, where $\vec{w} = (0, 1, 2)^T$.*

Exercise 57 *Let*

$$A_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

and let

$$A_2 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 0 \end{bmatrix}$$

Consider the linear transformation $\mathcal{T}_1 : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$\mathcal{T}_1([x, y, z]^T) = A_1 \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and consider the linear transformation $\mathcal{T}_2 : \mathbb{R}^2 \to \mathbb{R}^3$ *given by*

$$\mathcal{T}_2([x, y]^T) = A_2 \begin{bmatrix} x \\ y \end{bmatrix}.$$

Find the matrix A_3*, such that*

$$\mathcal{T}_2(\mathcal{T}_1([x, y, z]^T)) = A_3 \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Find $\mathcal{T}_2(\mathcal{T}_1([1, 2, 1]^T))$.

Exercise 58 *Give the augmented matrix corresponding to the following system of linear equations.*

$$\left\{ \begin{array}{rcrcrcr} 2x & + & 3y & + & 2z & = & 18 \\ x & - & y & - & z & = & 0 \\ x & + & y & + & z & = & 8 \end{array} \right.$$

Solve the system.

Exercise 59 *Give the augmented matrix corresponding to the following system of linear equations.*

$$\left\{ \begin{array}{rcrcrcrcr} x & + & 2y & + & 4z & - & u & = & 2 \\ & & y & - & z & & & = & 4 \\ 2x & + & 12y & + & z & + & u & = & 2 \\ 4x & + & 19y & + & 6z & - & u & = & 19 \end{array} \right.$$

Solve the system.

Exercise 60 *Give the augmented matrix corresponding to the following system of linear equations.*

$$\left\{ \begin{array}{rcrcrcr} x & + & 2y & + & 3z & = & 4 \\ & & 4y & - & z & = & 1 \\ 2x & + & 12y & + & 4z & = & 10 \end{array} \right.$$

Solve the system.

Exercise 61 *Give the augmented matrix corresponding to the following system of linear equations.*

$$\left\{ \begin{array}{rcrcrcrcr} x & + & 2y & + & 3z & + & u & = & 4 \\ & & 4y & - & z & + & u & = & 1 \\ 2x & + & 12y & + & 4z & + & 4u & = & 10 \end{array} \right.$$

Solve the system.

Exercise 62 *Let*

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 2 \end{bmatrix}$$

Determine A^{-1}. *Solve the equations* $A\vec{x} = [2,0,3]^T$, $A\vec{x} = [1,0,7]^T$, $A\vec{x} = [a,b,c]^T$, $A\vec{x} = [0,0,0]^T$. *Let* $\mathcal{T}$ *be the linear transformation corresponding to* A. *Determine* $Nullity(\mathcal{T})$ *and* $rank(\mathcal{T})$*?*

Exercise 63 *Consider*

$$A = \begin{bmatrix} 4 & 0 & 1 \\ 7 & 2 & 1 \\ 6 & 4 & 0 \end{bmatrix}$$

Solve the equation $A\vec{x} = \vec{0}$. *Let* $\mathcal{T}$ *be the linear transformation corresponding to* A. *What is* $N(\mathcal{T})$, $Nullity(\mathcal{T})$ *and* $rank(\mathcal{T})$*? Using elementary row operation on* A *derive a row echelon form (you might have done this already) and show that* $rank(\mathcal{T})$ *equals the number of non-zero rows. Is this a coincidence?*

Exercise 64 *Find the inverse of the following matrices*

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 2 & 0 & 5 \\ 0 & 3 & 6 \\ 0 & 0 & 4 \end{bmatrix}$$

Exercise 65 *Find (if possible) the inverse of the following matrix*

$$A = \begin{bmatrix} 4 & 7 & 3 \\ 1 & 2 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Exercise 66 *Let*

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} \\ a_{2,1} & a_{2,2} \end{bmatrix} \quad B = \begin{bmatrix} b_{1,1} & b_{1,2} \\ b_{2,1} & b_{2,2} \end{bmatrix}$$

Show by direct calculations that $det(AB) = det(A)det(B)$ *holds.*

Exercise 67 *Using the definition on page 123 determine*

$$det\left(\begin{bmatrix} 1 & 2 & 0 \\ 1 & 3 & 2 \\ 7 & 1 & 1 \end{bmatrix}\right)$$

Exercise 68 *Using a clever method determine* $det(A)$, *where*

$$A = \begin{bmatrix} 2 & 0 & 1 & -2 \\ 7 & 0 & 1 & 0 \\ 12 & 0 & 1 & 0 \\ 4 & 1 & 1 & 4 \end{bmatrix}$$

Does A *posses an inverse?*

Exercise 69 *Using a clever method determine* $det(A)$*, where*

$$A = \begin{bmatrix} 2 & 1 & 1 & 2 \\ 0 & 5 & 0 & 3 \\ 0 & 0 & 7 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Does A *posses an inverse?*

Exercise 70 *Find the volume of the parallelepiped bounded by basis vectors* $(0, 1, 2)$, $(2, 1, -1)$, $(4, 7, 0)$.

Exercise 71 *Using the method described at page 127 and page 128 determine* $det(A)$*, where*

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 4 & 1 & 1 & 0 \\ 5 & 1 & 2 & 7 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

4 Partial solutions to exercises for Chapter 3 of [VB]

Solution to Exercise 44:

1. Yes
2. $\{(-s, -s, s) \mid s \in \mathbf{R}\}$
3. 1,2

Solution to Exercise 58: $(x, y, z) = (4, 2, 2)$
Solution to Exercise 59: $\emptyset$
Solution to Exercise 60: $(x, y, z) = (3\frac{1}{2} - 3\frac{1}{2}s, \frac{1}{4} + \frac{1}{4}s, s)$, $s \in \mathbb{R}$
Solution to Exercise 65: Not possible

Solution to Exercise 67: 27

Solution to Exercise 68: 10. Yes.

Solution to Exercise 69: 70. Yes.

Solution to Exercise 71: -24.

5 Exercises for Chapter 4 of [VB]

Exercise 72 *Let* $V = \mathbb{R}^3$ *be described as an affine space by* $O = (0,0,0)^T$, $\{\vec{v}_0 = (1,0,0)^T, \vec{v}_1 = (0,1,0)^T, \vec{v}_2 = (0,0,1)^T\}$. *Let* $W = \mathbb{R}^3$ *be described as an affine space by* $O = (0,0,0)^T$, $\{\vec{w}_0 = (1,0,0)^T, \vec{w}_1 = (0,1,0)^T, \vec{w}_2 = (0,0,1)^T\}$. *Consider the affine transformation* $\mathcal{T} : V \to W$ *given by*

$$\mathcal{T}(O) = 2\vec{w}_0 + \vec{w}_1 - \vec{w}_2 + O$$

$$\mathcal{T}(\vec{v}_0) = \vec{w}_0 - \vec{w}_2$$

$$\mathcal{T}(\vec{v}_1) = \vec{w}_1 + \vec{w}_2$$

$$\mathcal{T}(\vec{v}_2) = \vec{w}_0 + \vec{w}_1 + 2\vec{w}_2$$

Describe the corresponding matrix (use the description at bottom of page 138). What is the transformation of the point $(1,2,1)^T$*?*

Exercise 73 *Let all values be as in exercise 72 with the only exception, that we replace the above values of* $\vec{w}_0$, $\vec{w}_1$ $\vec{w}_2$ *with the following:* $\vec{w}_0 = (1,0,1)^T$, $\vec{w}_1 = (0,1,0)^T$, $\vec{w}_2 = (0,1,1)^T$. *What is the the transformation of the point* $(1,2,1)^T$*?*

Exercise 74 *Consider the affine transformation* $\mathbb{R}^3 \to \mathbb{R}^3$ *corresponding to a translation of all points by* $\vec{t} = (1,-2,7)^T$. *Find the corresponding matrix.*

Exercise 75 *Find the matrix that describes the affine transformation* $\mathbb{R}^3 \to \mathbb{R}^3$ *given by a scaling along the* x*-axis with a factor* 2*, a scaling along the* y*-axis with a factor* 4*, and finally by a scaling along the* z*-axis with a factor* $1/3$*. What is the point* $(0,0,0)^T$ *transformed to? What is the point* $(1,1,1)^T$ *transformed to?*

Exercise 76 *This is a continuation of Exercise 74 and Exercise 75. Find the matrix that describes the affine transformation* $\mathbb{R}^3 \to \mathbb{R}^3$ *given by a scaling along the* x*-axis with a factor* 2*, a scaling along the* y*-axis with a factor* 4*, a scaling along the* z*-axis with a factor* $1/3$*, and finally a translation of all points by* $\vec{t} = (1,-2,7)^T$.

Exercise 77 *This is a continuation of Exercise 74 and Exercise 75. Find the matrix that describes the affine transformation* $\mathbb{R}^3 \to \mathbb{R}^3$ *given by a translation of all points by* $\vec{t} = (1,-2,7)^T$*, a scaling along the* x*-axis with a factor* 2*, a scaling along the* y*-axis with a factor* 4*, and finally a scaling along the* z*-axis with a factor* $1/3$.

Exercise 78 *Consider the inverse of the transformation treated in Exercise 74. Find the corresponding matrix.*

Exercise 79 *Consider the inverse of the transformation treated in Exercise 75. Find the corresponding matrix.*

Exercise 80 *How are the matrices in Exercise 76 and Exercise 77 related?*

Exercise 81 *Determine the matrix A that describes a rotation around the x-axis with -45^o. Find in an easy way A^{-1}.*

Exercise 82 *Determine the matrix A that describes the rotation around the z-axis with 45^o followed by a rotation around the y-axis with 180^o followed by a rotation around the x-axis with 90^o.*

Exercise 83 *Determine the matrix A that describes a rotation around the z-axis with -45^o followed by a rotation around the y-axis with -45^o followed by a rotation around the x-axis with 45^o. Find A^{-1}.*

Exercise 84 *Determine the matrix that describes rotation around the axis $(\sqrt{2}/2, \sqrt{2}/2, 0)^T$ with 45^o.*

Exercise 85 *Determine the matrix that describes rotation around the axis $(1, 1, 1)^T$ with 90^o (be careful!)*

Exercise 86 *Consider the affine transformation given by*

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 2 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the matrix that corresponds to the inverse affine transformation.

Exercise 87 *Consider the affine transformation given by*

$$\begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find the matrix that corresponds to the inverse affine transformation.

Exercise 88 *Consider the line in $\mathbb{R}^3$ given by*

$$L(t) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

Let Q_1 and Q_2 be two points on this line that are at distance $\sqrt{2}$ apart from each other. We now transform the affine by applying the transformation $\mathcal{T}$ given by:

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 2 & 0 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

What is the distance between $\mathcal{T}(Q_1)$ and $\mathcal{T}(Q_2)$?

Exercise 89 *Consider the matrix*

$$A = \begin{bmatrix} -\cos(\theta) & \sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have $AA^T = I$ *(please check); but* $det(A) = -1 \neq 1$ *(please check). Hence,* A *is not a rotation matrix. Give a physical explanation of what* A *is doing.*

Exercise 90 *Determine the matrix*

$$\begin{bmatrix} A & \vec{y} \\ \vec{0}^T & 1 \end{bmatrix}$$

that corresponds to a rotation with 90^o *around an axis through the point*

$$P_0 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}^T$$

and pointing in the direction

$$\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}^T$$

.

Exercise 91 *Consider the plane through origo with normal* $\vec{n} = [3, 4, 0]^T$. *We consider shear defined from this plane and from the shear vector* $\vec{s} = [-4, 3, 1]^T$. *Determine the matrix, that describes the shear as an affine transformation (be careful). Find the transformation of the point* $(1, 1, 1)^T$. *Find the transformation of the point* $(0, 0, 0)^T$.

Exercise 92 *Calculate the distance from the point* $(-23/5, 26/5, 12/5)^T$ *to the plane through origo with normal* $\hat{n} = [3/5, 4/5, 0]^T$. *Calculate the distance from* $(1, 1, 1)^T$ *to the same plane. The result hopefully indicates that your calculations in Exercise* **??** *are correct. Explain*

Exercise 93 *Consider the plane through the point* $(1, 1, 1)^T$ *with normal* $\vec{n} = [0, 4, 3]^T$. *We now perform a shear with respect to this plane and the shear vector* $\vec{s} = [2, 3, -4]^T$. *Determine the matrix that describes this shear. Find the transformation of the point* $(1, 1, 1)^T$. *Find the transformation of the point* $(0, 0, 0)^T$.

Exercise 94 *Let an affine transformation be given by the matrix*

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Find det(M) and M^{-1}. Consider the plane through $P = (1,1,1)^T$ with normal $\vec{n} = [1,0,0]^T$. Find the normal of the plane after the affine transformation. Find the transformation of the point P. Give the equation for the new plane.

Exercise 95 *Consider the reflection across the plane through origo with normal $\vec{n} = [0,3,4)^T$. Find the matrix that describes this transformation. Find the reflection of the point $(1,1,1)^T$.*

Exercise 96 *Consider the reflection through origo. Find the reflection of the point $(2,3,2)^T$.*

6 Partial solutions to exercises for Chapter 4 of [VB]

Solution to Exercise 74:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution to Exercise 75:

$$\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$(0,0,0)^T, (2,4,\frac{1}{3})^T$$

Solution to Exercise 76:

$$\begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 4 & 0 & -2 \\ 0 & 0 & \frac{1}{3} & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution to Exercise 77:

$$\begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 4 & 0 & -8 \\ 0 & 0 & \frac{1}{3} & \frac{7}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution to Exercise 82:

$$\begin{bmatrix} -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Solution to Exercise 83:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}-2}{4} & \frac{2-\sqrt{2}}{4} & -\frac{1}{2} \\ \frac{\sqrt{2}-2}{4} & \frac{2+\sqrt{2}}{4} & \frac{1}{2} \end{bmatrix}$$

$A^{-1} = A^T$.

Solution to Exercise 84:

$$\begin{bmatrix} \frac{\sqrt{2}+2}{4} & \frac{2-\sqrt{2}}{4} & \frac{1}{2} \\ \frac{2-\sqrt{2}}{4} & \frac{2+\sqrt{2}}{4} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}$$

Solution to Exercise 85: Hint: start by normalizing $\vec{r}$. If you forget to do so you will get a wrong result!

$$\begin{bmatrix} \frac{1}{3} & \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} \\ \frac{1+\sqrt{3}}{3} & \frac{1}{3} & \frac{1-\sqrt{3}}{3} \\ \frac{1-\sqrt{3}}{3} & \frac{1+\sqrt{3}}{3} & \frac{1}{3} \end{bmatrix}$$

Solution to Exercise 86:

$$\begin{bmatrix} \frac{1}{2} & 0 & 0 & -1 \\ 0 & \frac{1}{3} & 0 & -1 \\ 0 & 0 & \frac{1}{2} & -\frac{7}{2} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution to Exercise 90:

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \cdots$$

Solution to Exercise 91:

$$\begin{bmatrix} -7/5 & -16/5 & 0 & 0 \\ 9/5 & 17/5 & 0 & 0 \\ 3/5 & 4/5 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(-23/5, 26/5, 12/5)$, $(0, 0, 0)$

Solution to Exercise 92: 7/5, 7/5. Use the fact that $\vec{s}$ is contained in the plane.

Solution to Exercise 93:

$$\begin{bmatrix} 1 & 8/5 & 6/5 & -14/5 \\ 0 & 17/5 & 9/5 & -21/5 \\ 0 & -16/5 & -7/5 & 28/5 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(1, 1, 1)$, $(-14/5, -21/5, 28/5)$

Solution to Exercise 94:

$$M^{-1} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\vec{n}' = [1, 0, 0]^T$, $(4, 3, 2)$, $x - 4 = 0$.

Solution to Exercise 95:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 7/25 & -24/25 & 0 \\ 0 & -24/25 & -7/25 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$(1, -17/25, -31/25)$

Solution to Exercise 96: You answered the exercise correctly...

7 Exercises for Chapter 5 of [VB]

Exercise 97 *Consider the matrix*

$$R = \begin{bmatrix} \frac{\sqrt{6}}{4} & -\frac{\sqrt{6}}{4} & \frac{1}{2} \\ \frac{\sqrt{6}+2\sqrt{2}}{8} & \frac{-\sqrt{6}+2\sqrt{2}}{8} & -\frac{3}{4} \\ \frac{-\sqrt{2}+2\sqrt{6}}{8} & \frac{\sqrt{2}+2\sqrt{6}}{8} & -\frac{\sqrt{3}}{4} \end{bmatrix}$$

Find angles $\theta_x, \theta_y, \theta_z$ such that $R = R_x R_y R_z$. How many choices are there of $\theta_x, \theta_y, \theta_z$? List them.

Exercise 98 *Consider the matrix*

$$R = \begin{bmatrix} 0 & 0 & 1 \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Find angles $\theta_x, \theta_y, \theta_z$ such $R = R_x R_y R_z$. Are there more choices of $\theta_x, \theta_y, \theta_z$?

Exercise 99 *Consider the matrix*

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

1. *Show that A is a rotation matrix.*
2. *What happens to the point $[-1, -1, 0]^T$ under the rotation?*
3. *Determine rotation angle and rotation axis.*

Exercise 100 *Consider the rotation matrix*

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Determine rotation angle and rotation axis.

Exercise 101 *Find the quaternion that corresponds to rotation around the axis $[1, 2, 2]^T$ with $\theta = 90^o$.*

Exercise 102 *Determine the quaternion that corresponds to rotation around the axis $[-1, -2, -2]^T$ with $\theta = 270^o$.*

Exercise 103 *Determine the quaternion that corresponds to rotation around the axis $[1, 1, 1]^T$ with $\theta = 60^o$.*

Exercise 104 *Determine the quaternion that corresponds to the rotoation matrix A from Exercise 99 in the following two ways*

1. *Use the result from Exercise 99*
2. *Use the theory from page 191*

Exercise 105 *Determine the quaternion that corresponds to the matrix A from Exercise 100 in the following two ways*

1. *Use the result from Exercise 100*
2. *Use the theory from page 191*

Exercise 106 *Given the quaternion $q = \frac{\sqrt{2}}{2} + (\frac{\sqrt{2}}{2}, 0, 0)$ find all possible corresponding fixed angles.*

Exercise 107 *Given the quaternion $q = \frac{\sqrt{2}}{2} + (\frac{1}{2}, \frac{1}{2}, 0)$*

1. *Show that* $\| q \|= 1$.

2. *Determine the corresponding rotation matrix*

Exercise 108 *Determine the quaternion* q_z *that corresponds to rotation around the* z*-axis with* θ_z. *Determine the quaternion* q_y *that corresponds to rotation around the* y*-axis with* θ_y. *Determine the quaternion* q_x *that corresponds to rotation around the* x*-axis with* θ_x. *Calculate* $q_x q_y q_z$ *and compare to the theory at the top of page 193.*

Exercise 109 *Consider the quaternions* $p = 2 + (1, 0, 1)$ *and* $q = 1 + (1, 3, 0)$.

1. *Find* pq

2. *Find* p^{-1}

Exercise 110 *Find the quaternion* q *that corresponds to rotation around the axis* $(1, 1, 0)$ *with* 90^o. *Consider the point* $(1, 1, 1)$. *Let* $p_1 = 0 + (1, 1, 1)$ *be the corresponding quaternion. Calculate* qp_1q^{-1}. *Let* $p_2 = 0 + (1, 1, 0)$. *Argue that* qp_2q^{-1} *equals* p_2. *Show this by performing the exact calculations.*

Exercise 111 *Consider the rotation with rotation axis* $(1, 0, 1)$ *and rotation angle* 90^o. *Given the point* (x, y, z) *consider the quaternion* $p = 0 + (x, y, z)$. *The above rotation sends* (x, y, z) *to* (x', y', z') *corresponding to the quaternion* $p' = 0 + (x', y', z')$.

1. *Determine the quaternionen* q *such that* $p' = qpq^{-1}$

2. *Find the matrix* A *such that*

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Exercise 112 *Consider the rotation with rotation axis* $(4, 0, 3)$ *and rotation angle* 90^o. *Given the point* (x, y, z) *consider the quaternion* $p = 0 + (x, y, z)$. *The above rotation sends* (x, y, z) *to* (x', y', z') *corresponding to the quaternion* $p' = 0 + (x', y', z')$.

1. *Determine the quaternionen* q *such that* $p' = qpq^{-1}$.

2. *Demonstrate that* $\| q \|= 1$

3. *What is* $(1, 0, 0)$ *being rotated to?*

8 Partial solutions to exercises for Chapter 5 of [VB]

Solution to Exercise 97: $(120^o, 30^o, 45^o)$, $(300^o, 150^o, 225^o)$.

Solution to Exercise 98: One possibility is $(45^o, 90^o, 0^o)$. This is not the only possibility.

Solution to Exercise 99: $[-1,-1,0]^T$, 90^o, $[1,1,0]^T$ (your answer may differ by a positive scaling factor).

Solution to Exercise 100: 180^o, $[1,0,1]^T$ (your answer may differ by a positive scaling factor)

Solution to Exercise 101: $\frac{\sqrt{2}}{2} + (\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3})$.

Solution to Exercise 102: $-\frac{\sqrt{2}}{2} - (\frac{\sqrt{2}}{6}, \frac{\sqrt{2}}{3}, \frac{\sqrt{2}}{3})$.

Solution to Exercise 103: $\frac{\sqrt{3}}{2} + (\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}})$.

Solution to Exercise 104: $\frac{\sqrt{2}}{2} + (\frac{1}{2}, \frac{1}{2}, 0)$.

Solution to Exercise 105: $0 + (\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2})$.

Solution to Exercise 106: $(\theta_x, \theta_y, \theta_z) = (90^o, 0^o, 0^o)$ og $(\theta_x, \theta_y, \theta_z) = (90^o, 180^o, 180^o)$

Solution to Exercise 107:

$$\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

Solution to Exercise 108: $q_z = \cos(\theta_z/2) + \sin(\theta_z/2)(0,0,1)$, $q_y = \cos(\theta_y/2) + \sin(\theta_y/2)(0,1,0)$, $q_x = \cos(\theta_x/2) + \sin(\theta_x/2)(1,0,0)$.

Solution to Exercise 109: $1 + (0,7,4)$, $1/3 - (1/6, 0, 1/6)$.

Solution to Exercise 110: $\frac{\sqrt{2}}{2} + (\frac{1}{2}, \frac{1}{2}, 0)$, $(\frac{2+\sqrt{2}}{2}, \frac{2-\sqrt{2}}{2}, 0)$.

Solution to Exercise 111: $\frac{\sqrt{2}}{2} + (\frac{1}{2}, 0, \frac{1}{2})$

$$\begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ \frac{\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} \\ \frac{1}{2} & \frac{\sqrt{2}}{2} & \frac{1}{2} \end{bmatrix}$$

Solution to Exercise 112: $\frac{\sqrt{2}}{2} + (\frac{4\sqrt{2}}{10}, 0, \frac{3\sqrt{2}}{10})$, $(64/100, 60/100, 48/100)$.

9 Exercises for Chapter 10 of [VB]

Exercise 113 *In this exercise we use linear interpolation to fit a curve $P(t)$ through the points $P_0 = (1, 0, 1)$ og $P_1 = (-1, 2, -7)$. We want to pass P_0 at time $t_0 = 2$ and to pass P_1 at time $t_1 = 10$. Which point do the curve pass at time $t = 4$? Determine $P(t)$.*

Exercise 114 *Find the Hermite curve $Q(u)$, $u = 0, \ldots, 1$ with $Q(0) = (0, 1, 2)$, $Q'(0) = (2, 1, 0)$, $Q(1) = (2, 2, 2)$ and $Q'(1) = (2, 1, 0)$. Explain the nice answer.*

Exercise 115 *Find the Hermite curve $Q(u)$, $u = 0, \ldots, 1$ with $Q(0) = (0, 1, 2)$, $Q'(0) = (1, 0, 0)$, $Q(1) = (2, 2, 2)$ and $Q'(1) = (1, 0, 1)$.*

Exercise 116 *Find the piecewise Hermite curve $Q_0(u), Q_1(u)$ such that $Q_0(0) = (0, 0, 0)$, $Q_0'(0) = (1, 0, 0)$, $Q_0(1) = Q_1(0) = (1, 1, 0)$, $Q_0'(1) = Q_1'(0) = (0, 0, 1)$, $Q_1(1) = (1, 0, 0)$, $Q_1'(1) = (0, 1, 0)$ hold.*

Exercise 117 *Find the piecewise Hermite curve $Q_0(u), Q_1(u)$ such that $Q_0(0) = (0, 0, 0)$, $Q_0'(0) = (1, 0, 0)$, $Q_0(1) = Q_1(0) = (1, 1, 0)$, $Q_1(1) = (1, 0, 0)$, $Q_1'(1) = (0, 1, 0)$, $Q_0'(1) = Q_1'(0)$ and $Q_0''(1) = Q_1''(0)$ hold.*

Exercise 118 *Find the piecewise Hermite curve $Q_0(u), Q_1(u)$ such that $Q_0(0) = (0, 0, 0)$, $Q_0(1) = Q_1(0) = (1, 1, 0)$, $Q_1(1) = (1, 0, 0)$, $Q_0'(1) = Q_1'(0)$, $Q_0''(1) = Q_1''(0)$ and with natural end conditions.*

Exercise 119 *Consider the Bezier curve with control points $P_0 = (1, 0, 1)$, $P_1 = (2, 0, 1)$, $P_2 = (2, 2, 3)$ and $P_3 = (2, 2, 2)$. Find $Q(u)$. What are the tangent to the curve at the end points P_0 and P_3?*

Exercise 120 *Consider the quaternions $p = \frac{\sqrt{2}}{2} + (0, \frac{\sqrt{2}}{2}, 0)$ and $q = \frac{\sqrt{2}}{2} + (\frac{\sqrt{2}}{2}, 0, 0)$. Determine the function slerp(p, q, t). Calculate (in floating point) slerp$(p, q, \frac{1}{4})$.*

Exercise 121 *Let the quaternions p and q be as in Exercise 120. Calculate (in floating point) lerp$(p, q, \frac{1}{4})$*

10 Solutions to exercises for Chapter 10

Solution to Exercise 113: $P(t) = (3/2, -1/2, 3) + t(-1/4, 1/4, -1)$, $P(4) = (1/2, 1/2, -1)$.
Solution to Exercise 114: $Q(u) = u(2, 1, 0) + (0, 1, 2)$.

Solution to Exercise 115: $Q(u) = u^3(-2, -2, 1) + u^2(3, 3, -1) + u(1, 0, 0) + (0, 1, 2)$.

Solution to Exercise 116: $Q_0(u) = u^3(-1,-2,1) + u^2(1,3,-1) + u(1,0,0) + (0,0,0)$, $Q_1(u) = u^3(0,3,1) + u^2(0,-4,-2) + u(0,0,1) + (1,1,0)$

Solution to Exercise 117: (mellemresultat: $\vec{P}_1' = (1/2,-1/4,0)$). $Q_0(u) = u^3(-1/2,-9/4,0)+u^2(1/2,13/4,0)+u(1,0,0)+(0,0,0)$, $Q_1(u) = u^3(1/2,11/4,0)+u^2(-1,-7/2,0) + u(1/2,-1/4,0) + (1,1,0)$

Solution to Exercise 118: (mellemresultat: $\vec{P}_0' = (5/4,3/2,0)$, $\vec{P}_1' = (1/2,0,0)$, $\vec{P}_2' = (-1/4,-3/2,0)$) $Q_0(u) = u^3(-1/4,-1/2,0) + u(5/4,3/2,0) + (0,0,0)$, $Q_1(u) = u^3(1/4,1/2,0) + u^2(-3/4,-3/2,0) + u(1/2,0,0) + (1,1,0)$

Solution to Exercise 119: $(u^3 - 3u^2 + 3u + 1, 4u^3 + 6u^2, -5u^3 + 6u^2 + 1)$. $3(P_1 - P_0) = (3,0,0)$, $3(P_3 - P_2) = (0,0,-3)$.

Solution to Exercise 120:

$$\frac{\sin((1-t)60^o)p + \sin(t60^o)q}{\frac{\sqrt{3}}{2}}$$

$$(0.7887, 0.2113, 0.5774, 0)$$

Solution to Exercise 121:

$$(0.7845, 0.1961, 0.5883, 0)$$