MCG - 2

Operations on vectors:

Vectoraddition: if $\mathbf{v}$ and $\mathbf{w}$ are vectors then $\mathbf{v}+\mathbf{w}$ is a vector.

$$(v_0, v_1, \ldots, v_{n-1}) + (w_0, w_1, \ldots, w_{n-1}) = (v_0+w_0, v_1+w_1, \ldots, v_{n-1}+w_{n-1}).$$

Scalarmultiplication: if $\mathbf{v}$ is a vector and a is a number (scalar) then $a\mathbf{v}$ is a vector.

$$a(v_0, v_1, \ldots, v_{n-1}) = (av_0, av_1, \ldots, av_{n-1}).$$

Usual algebraic laws are valid for these operations.
E.g. $1\mathbf{v} = \mathbf{v}$ og $0\mathbf{v} = \mathbf{0}$.

If $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ are vectors and $a_0, a_1, \ldots, a_{n-1}$ are numbers then the expression

$$a_0\mathbf{v_0} + a_1\mathbf{v_1} + \ldots + a_{n-1}\mathbf{v_{n-1}}$$

is called a linear combination of $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$.

The set of vectors that are can be written as linear combinations of $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ is called the set (or subspace) spanned by $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$.

If one of the n vectors $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ can be written as a linear combination of the other $n-1$ vectors then the vectors are said to be linearly dependent. Otherwise they are linearly independent.

The dotproduct of two vectors $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$ and $\mathbf{w} = (w_0, w_1, \ldots, w_{n-1})$ is defined by

$$\mathbf{v} \cdot \mathbf{w} = v_0 w_0 + v_1 w_1 + \ldots + v_{n-1} w_{n-1}.$$

The dotproduct also satisfies

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \; ||\mathbf{w}|| \cos \theta$$

where θ is the angle between the vectors.

$\mathbf{v}$ and $\mathbf{w}$ are orthogonal if $\mathbf{v} \cdot \mathbf{w} = 0$.

The length of $\mathbf{v}$ is $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_0^2 + v_1^2 + \ldots + v_{n-1}^2}$.

The dotprodduct satisfies the following laws:

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $a(\mathbf{v} \cdot \mathbf{w}) = (a\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (a\mathbf{w})$
- $\mathbf{v} \cdot \mathbf{v} \geq 0$ and
- $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

The length of vectors satisfies:

- $||\mathbf{v}|| \geq 0 \quad$ and $\quad ||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

- $||a\mathbf{v}|| = |a| \; ||\mathbf{v}||$

- $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.

These laws are also satisfied by the Manhattan norm

$$||\mathbf{v}||_{\ell_1} = |v_0| + |v_1| + \ldots + |v_{n-1}|$$

where $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$.

Normalizing a vector $\mathbf{v} \neq \mathbf{0}$:

$$\hat{\mathbf{v}} = \frac{1}{||\mathbf{v}||}\mathbf{v}.$$

$\hat{\mathbf{v}}$ has the same direction as $\mathbf{v}$ and it has length 1.

The projection of a vector $\mathbf{v}$ on a vector $\mathbf{w} \neq \mathbf{0}$ er

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w} = (\mathbf{v} \cdot \hat{\mathbf{w}})\hat{\mathbf{w}}.$$

The vector

$$\text{perp}_{\mathbf{w}}\mathbf{v} = \mathbf{v} - \text{proj}_{\mathbf{w}}\mathbf{v}$$

is orthogonal to $\mathbf{w}$.

A set of vectors $\{\mathbf{w_0}, \mathbf{w_1}, \ldots, \mathbf{w_{n-1}}\}$ is said to be orthonormal if the vectors are orthogonal and have length 1.

Gram-Schmidt orthogonalization of linearly independent vectors $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$:

- $\mathbf{w_0} = \mathbf{v_0}$
- $\mathbf{w_1} = \mathbf{v_1} - \text{proj}_{\mathbf{w_0}}\mathbf{v_1}$
- $\mathbf{w_2} = \mathbf{v_2} - \text{proj}_{\mathbf{w_0}}\mathbf{v_2} - \text{proj}_{\mathbf{w_1}}\mathbf{v_2}$
- $\ldots$

In general:

$$\mathbf{w_i} = \mathbf{v_i} - \mathsf{proj}_{\mathbf{w_0}}\mathbf{v_i} - \ldots - \mathsf{proj}_{\mathbf{w_{i-1}}}\mathbf{v_i}.$$

Finally compute

$$\hat{\mathbf{w}}_{\mathbf{0}}, \hat{\mathbf{w}}_{\mathbf{1}}, \ldots, \hat{\mathbf{w}}_{\mathbf{n-1}}.$$

These vectors are orthonormal.

MCG - 3

$\mathbf{u}, \mathbf{v}, \mathbf{w}$: three linearly independent vectors in $\mathbb{R}^3$.

Use right hand:
index finger points in direction $\mathbf{u}$
middle finger points in direction $\mathbf{v}$.

Then we say that $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is right-handed if $\mathbf{w}$ is on the same side of the plane spanned by $\mathbf{u}, \mathbf{v}$ as the thumb.

Otherwise $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is left-handed.

Example: $\mathbf{i}, \mathbf{j}, \mathbf{k}$ is right-handed.

Let $\mathbf{v} = (v_x, v_y, v_z)$ and $\mathbf{w} = (w_x, w_y, w_z)$.

Then the cross product is defined by

$$\mathbf{v} \times \mathbf{w} = (v_y w_z - w_y v_z, v_z w_x - w_z v_x, v_x w_y - w_x v_y).$$

$\mathbf{v} \times \mathbf{w}$ is the vector orthogonal to $\mathbf{v}$ and $\mathbf{w}$, satisfying that:

$\mathbf{v}, \mathbf{w}, \mathbf{v} \times \mathbf{w}$ is right-handed and

$||\mathbf{v} \times \mathbf{w}|| = ||\mathbf{v}|| \; ||\mathbf{w}|| \sin\theta$,

where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

$||\mathbf{v} \times \mathbf{w}||$ is the area of a parallelogram where $\mathbf{v}$ and $\mathbf{w}$ are two edges.

Vector triple product:

If $\mathbf{v}$ and $\mathbf{w}$ are two vectors in $\mathbb{R}^3$ (non-parallel) then

$$\mathbf{w},\ \mathbf{v}\times\mathbf{w},\ \mathbf{w}\times(\mathbf{v}\times\mathbf{w})$$

is a right-handed orthogonal basis.
(Alternative to Gram-Schmidt.)

Scalar triple product:

$$\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \mathbf{w}\cdot(\mathbf{u} \times \mathbf{v}) = \mathbf{v}\cdot(\mathbf{w} \times \mathbf{u})$$

is a number which is

- positive if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is right-handed,
- negative if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ is left-handed,
- 0 if $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly dependent.

$|\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w})|$ is the volume (rumfang) of a parallelopiped where $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are three edges.

If V is a set of vectors in $\mathbb{R}^n$ satisfying

- $\mathbf{v} \in V$ and $\mathbf{w} \in V \Rightarrow \mathbf{v} + \mathbf{w} \in V$.

- $\mathbf{v} \in V$ and $c \in \mathbb{R} \Rightarrow c\mathbf{v} \in V$.

the we say that V is a subspace of $\mathbb{R}^n$.

If $\mathbf{b}_1, \ldots, \mathbf{b}_d$ are linearly independent vectors spanning V then we say that $\{\mathbf{b}_1, \ldots, \mathbf{b}_d\}$ is a basis for V.
d is then the dimensionen of V.

Subspace of dimension 0: $\{\mathbf{0}\}$
Subspace of dimension 1: line through $\mathbf{0}$.
Subspace of dimension 2: plane through $\mathbf{0}$.

Affine space of dimension 1: line (not through $\{\mathbf{0}\}$).
Affine space of dimension 2: plan (not through $\{\mathbf{0}\}$).

An affine space consists of points on the form

$$O + \mathbf{v}, \quad \mathbf{v} \in V,$$

where V is a subspace and O is a fixed point.

P_0 and P_1: two different points.
There is a unique line passing through both points. It consists of points on the form

$$tP_0 + (1-t)P_1, \quad t \in \mathbb{R}.$$

The line segment between P_0 and P_1 consists of points

$$tP_0 + (1-t)P_1, \quad \text{hvor } 0 \leq t \leq 1.$$

A set of points is said to be convex if for every pair of points P_0, P_1 in the set, the line segment between them is also contained in the set.

Let $P_0, \ldots, P_k$ be points.
The expression

$$a_0 P_0 + a_1 P_1 + \ldots + a_k P_k, \quad \text{where } a_0 + a_1 + \ldots + a_k = 1$$

is called an affine combination of $P_0, \ldots, P_k$.

The set of points that can be written as an affine combination of $P_0, \ldots P_k$ is an affine space.

$P_0, \ldots, P_k$ are said to be affinely dependent if one of the points can be written as an affine combination of the other points.
Otherwise $P_0, \ldots, P_k$ are affinely independent.

W: an affine space, $P_0, \ldots, P_k \in W$.
If every point in W is an affine combination of $P_0, \ldots, P_k$ and if these points are affinely independent then we say that $P_0, \ldots, P_k$ is a simplex.

Every point P in W can then be written (in one and only one way) as

$$a_0 P_0 + a_1 P_1 + \ldots + a_k P_k, \quad \text{where } a_0 + a_1 + \ldots + a_k = 1.$$

$a_0, a_1, \ldots, a_k$ are called the barycentric coordinates for P.

MCG - 4

The polar coordinates for a point (x, y) in the plane is (r, θ) where $r = \sqrt{x^2 + y^2}$ is the distance from $(0, 0)$ to (x, y), and θ is the angel (in positive direction) from the x-axis to the vector (x, y).

Converting from (r, θ) to (x, y):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Converting from (x, y) to (r, θ):

$$r = \sqrt{x^2 + y^2}, \quad \theta = \begin{cases} \arctan\frac{y}{x} & \text{hvis } x > 0, \\ \arctan\frac{y}{x} + \pi & \text{hvis } x < 0, \\ \frac{\pi}{2} & \text{hvis } x = 0, y > 0, \\ -\frac{\pi}{2} & \text{hvis } x = 0, y < 0. \end{cases}$$

If you prefer to work with degrees then replace π by 180°.

The spherical coordinates for a point $P = (x, y, z)$ in space are (ρ, ϕ, θ) where $\rho = \sqrt{x^2 + y^2 + z^2}$ is the distance from $(0, 0, 0)$ to (x, y, z), and ϕ is the angel between the z-axis and the vector (x, y, z).
$0 \le \phi \le \pi$ (or $0 \le \phi \le 180^\circ$). θ is the same as in polar coordinates for (x, y).

Converting from (ρ, ϕ, θ) to (x, y, z):

$$x = \rho \sin\phi \cos\theta, \quad y = \rho \sin\phi \sin\theta, \quad z = \rho \cos\phi.$$

Converting from (x, y, z) to (ρ, ϕ, θ):

$$\rho = \sqrt{x^2 + y^2 + z^2}, \quad \phi = \arccos\frac{z}{\rho}.$$

θ is computed as on the previous page.

A line passing through points P_0 and P_1 consisits of points that can be written in parametric form as

$$P_0 + t\mathbf{d}, \quad t \in \mathbb{R},$$

where $\mathbf{d} = P_1 - P_0$ is the vector from P_0 to P_1.

For a line in the plane there a vector $\mathbf{n} = (a, b)$ (e.g. if $\mathbf{d} = (b, -a)$) perpendicular to the line.

A point $Q = (x, y)$ lies on the line if and only if

$$\mathbf{n} \cdot (Q - P_0) = 0.$$

If $P_0 = (x_0, y_0)$ then this equation can be written as

$$ax + by + c = 0,$$

where $c = -ax_0 - by_0$. This is called a generalized line equation.

If $||\mathbf{n}|| = \sqrt{a^2 + b^2} = 1$ and $ax + by + c = d$ then the point (x, y) is in distance $|d|$ from the line – if $d > 0$ on the same side of the line as indicated by $\mathbf{n}$.

A plane passing through the points P_0, P_1, P_2 consists of points that can be written in parametric form as

$$P_0 + s\mathbf{u} + t\mathbf{v}, \qquad s, t \in \mathbb{R},$$

where $\mathbf{u} = P_1 - P_0$ and $\mathbf{v} = P_2 - P_0$.

For a plane in $\mathbb{R}^3$ there is a vector $\mathbf{n} = (a, b, c)$ (e.g. $\mathbf{n} = \mathbf{u} \times \mathbf{v}$) perpendicular to the plane.

A point $Q = (x, y, z)$ lies on the plane if and only if

$$\mathbf{n} \cdot (Q - P_0) = 0.$$

If $P_0 = (x_0, y_0, z_0)$ then this equation can be written as

$$ax + by + cz + d = 0,$$

where $d = -ax_0 - by_0 - cz_0$. This is called a generalized plane equation.

If $||\mathbf{n}|| = \sqrt{a^2 + b^2 + c^2} = 1$ and (x, y, z) is an arbitrary point in space then $|ax + by + cz + d|$ is the distance between the point and the plane – if $ax + by + cz + d > 0$ on the same side of the plane as indicated by $\mathbf{n}$.

Let P be a point on the plane passing through P_0, P_1, P_2.
Then there exists unique numbers s, t so that

$$P = P_0 + s\mathbf{u} + t\mathbf{v}, \quad \text{where } \mathbf{u} = P_1 - P_0 \text{ and } \mathbf{v} = P_2 - P_0.$$

If $\mathbf{w} = P - P_0 = s\mathbf{u} + t\mathbf{v}$ then s and t can be determined from the equations

$$\mathbf{v} \times \mathbf{w} = s(\mathbf{v} \times \mathbf{u}), \quad \mathbf{u} \times \mathbf{w} = t(\mathbf{u} \times \mathbf{v}).$$

Then

$$P = P_0 + s(P_1 - P_0) + t(P_2 - P_0) = (1 - s - t)P_0 + sP_1 + tP_2.$$

Thus the barycentric coordinates for P are $(1 - s - t, s, t)$.

If P is inside the triangle with vertices P_0, P_1, P_2 then $1 - s - t \geq 0, s \geq 0, t \geq 0$.
If one of the numbers is negative then P is outside the triangle.

MCG - 5

A 3×5 matrix:

$$A = \begin{bmatrix} 1 & 0 & 2 & 1 & -1 \\ 3 & 2 & 7 & -5 & 0 \\ 1 & 1 & 2 & 1 & 4 \end{bmatrix}.$$

An $m \times n$ matrix has m rows, and n columns.
Rows are enumerated $0, 1, \ldots, m-1$.
Columns are enumerated $0, 1, \ldots, n-1$.

The element (number) in row i, column j is written $(A)_{ij}$ or a_{ij}.
In the example: $(A)_{12} = 7$.

If A and B are $m \times n$ matrices then $A + B$ is the $m \times n$ matrix where $(A + B)_{ij} = (A)_{ij} + (B)_{ij}$.

If A is an $m \times n$ matrix and $a \in \mathbb{R}$ is a number then aA is the $m \times n$ matrix where $(aA)_{ij} = a(A)_{ij}$.

A an $m \times n$ matrix.
B an $r \times s$ matrix.

The product AB exists if $n = r$
and then the result is an $m \times s$ matrix.

$$\begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \\ 5 & 6 & 7 & 8 \\ * & * & * & * \end{bmatrix} \begin{bmatrix} * & 1 & * & * & * & * \\ * & 2 & * & * & * & * \\ * & 3 & * & * & * & * \\ * & 4 & * & * & * & * \end{bmatrix} = \begin{bmatrix} * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \\ * & 70 & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}$$

$70 = 5 \cdot 1 + 6 \cdot 2 + 7 \cdot 3 + 8 \cdot 4.$

Algebraic rules, a few examples:

$$A(B+C) = AB + AC$$

and

$$A(aB) = a(AB),$$

where a is a number and A, B, C are matrices with sizes so that the addition and multiplication is defined.

Almost all usual algebraic rules are satisfied.
Except that multiplication is not commutative:

$$AB \neq BA.$$

The transposed of an $m \times n$ matrix A is an $n \times m$ matrix A^T where $(A^T)_{ij} = A_{ji}$.

If

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \end{bmatrix}$$

then

$$A^T = \begin{bmatrix} 1 & 5 & 9 \\ 2 & 6 & 10 \\ 3 & 7 & 11 \\ 4 & 8 & 12 \end{bmatrix}.$$

$$(A + B)^T = A^T + B^T, \quad (AB)^T = B^T A^T.$$

Identity matrix:

$$I = I_n = I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

If A is an $m \times n$ matrix then $AI_n = A$ and $I_m A = A$.

An $n \times 1$ matrix is a (column) vector.

A $1 \times n$ matrix is a (row) vektor. It is written as the transposed of a column vector.

Product of block matrices (if all sums and products are defined):

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}.$$

$$\begin{bmatrix} \mathbf{a}_0 & \ldots & \mathbf{a}_{n-1} \end{bmatrix} \begin{bmatrix} b_0 \\ \vdots \\ b_{n-1} \end{bmatrix} = b_0 \mathbf{a}_0 + \ldots + b_{n-1} \mathbf{a}_{n-1}.$$

$$A \begin{bmatrix} \mathbf{b}_0 & \ldots & \mathbf{b}_{n-1} \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_0 & \ldots & A\mathbf{b}_{n-1} \end{bmatrix}.$$

Let V and W be vector space, e.g. $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$.

A function $T : V \mapsto W$ is said to be a linear transformation if

- $T(\mathbf{v} + \mathbf{w}) = T(\mathbf{v}) + T(\mathbf{w})$ for all vectors $\mathbf{v}, \mathbf{w} \in V$, and

- $T(a\mathbf{v}) = aT(\mathbf{v})$ for all vectors $\mathbf{v} \in V$ and all numbers a.

Example. Let $\mathbf{v} = [v_x, v_y, v_z]^T \in \mathbb{R}^3$.
Then $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ defined by $T(\mathbf{x}) = \mathbf{v} \times \mathbf{x}$ is a linear transformation and $T(\mathbf{x}) = \tilde{\mathbf{v}}\mathbf{x}$ where $\tilde{\mathbf{v}}$ is the 3×3 matrix

$$\tilde{\mathbf{v}} = \begin{bmatrix} 0 & -v_z & v_y \\ v_z & 0 & -v_x \\ -v_y & v_x & 0 \end{bmatrix}.$$

Example. Let $\hat{\mathbf{v}} =\in \mathbb{R}^n$, with $||\hat{\mathbf{v}}|| = 1$.
Then $T : \mathbb{R}^n \mapsto \mathbb{R}^n$ defined by $T(\mathbf{x}) = \text{proj}_{\hat{\mathbf{v}}}\mathbf{x} = (\mathbf{x}\cdot\hat{\mathbf{v}})\hat{\mathbf{v}}$ is a linear transformation and $T(\mathbf{x}) = A\mathbf{x}$ where A is the $n \times n$ matrix

$$A = \hat{\mathbf{v}}\hat{\mathbf{v}}^T = (\hat{\mathbf{v}} \otimes \hat{\mathbf{v}}).$$

In general

$$(\mathbf{v} \otimes \mathbf{w}) = \mathbf{v}\mathbf{w}^T$$

is called a tensor product.

MCG - 6

If A is an $m \times n$ matrix then the function

$$\mathcal{S} : \mathbb{R}^n \mapsto \mathbb{R}^m$$

defined by

$$\mathcal{S}(\mathbf{v}) = A\mathbf{v}$$

is a linear transformation.

If $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation, satisfying

$$\mathcal{T}(\mathbf{e}_0) = \mathbf{a}_0,\ \mathcal{T}(\mathbf{e}_1) = \mathbf{a}_1,\ \ldots,\ \mathcal{T}(\mathbf{e}_{n-1}) = \mathbf{a}_{n-1},$$

where

$$\mathbf{e}_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_{n-1} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

then

$$\mathcal{T}(\mathbf{v}) = A\mathbf{v},$$

where A is the matrix

$$\begin{bmatrix} \mathbf{a}_0 \ \mathbf{a}_1 \ldots \mathbf{a}_{n-1} \end{bmatrix}.$$

If the linear transformation $\mathcal{S} : \mathbb{R}^p \mapsto \mathbb{R}^m$ satifies $\mathcal{S}(\mathbf{w}) = A\mathbf{w}$

and the linear transformation $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^p$ satisfies $\mathcal{T}(\mathbf{v}) = B\mathbf{v}$

then

$$\mathcal{S} \circ \mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$$

is also a linear transformation and

$$(\mathcal{S} \circ \mathcal{T})(\mathbf{v}) = (AB)\mathbf{v}.$$

If $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a linear transformation then we define the null space of $\mathcal{T}$ as

$$N(\mathcal{T}) = \{\mathbf{v} \in \mathbb{R}^n \mid \mathcal{T}(\mathbf{v}) = \mathbf{0}\}.$$

This is a subspace of $\mathbb{R}^n$.
The dimensionen of $N(\mathcal{T})$ is called *nullity*$(\mathcal{T})$.

The range of $\mathcal{T}$ is

$$R(\mathcal{T}) = \{\mathbf{w} \in \mathbb{R}^m \mid \text{there exists } \mathbf{v} \in \mathbb{R}^n \text{ so that } \mathcal{T}(\mathbf{v}) = \mathbf{w}\}.$$

This is a subspace of $\mathbb{R}^m$.
The dimension of $R(\mathcal{T})$ is called the rank of $\mathcal{T}$ and is written as *rank*$(\mathcal{T})$.

The dimensions satisfy the following equation:

$$\textit{nullity}(\mathcal{T}) + \textit{rank}(\mathcal{T}) = n.$$

A system of linear equations

$$
\begin{array}{rcl}
a_{00}x_0 + & a_{01}x_1 + \ldots + a_{0,n-1}x_{n-1} = & b_0 \\
a_{10}x_0 + & a_{11}x_1 + \ldots + a_{1,n-1}x_{n-1} = & b_1 \\
\vdots & & \\
a_{m-1,0}x_0 + & a_{m-1,1}x_1 + \ldots + a_{m-1,n-1}x_{n-1} = & b_{m-1}
\end{array}
$$

can be denoted by its augmented coefficient matrix

$$
\begin{bmatrix}
a_{00} & a_{01} & \cdots & a_{0,n-1} & b_0 \\
a_{10} & a_{11} & \cdots & a_{1,n-1} & b_1 \\
\vdots & \vdots & & \vdots & \vdots \\
a_{m-1,0} & a_{m-1,1} & \cdots & a_{m-1,n-1} & b_{m-1}
\end{bmatrix}
$$

Elementary row operations on matrices:

1. multiply a row by a number $k \neq 0$
2. replace row i by (row i) $+ k \cdot$ (row j), $i \neq j$
3. swap two rows.

Two $m \times n$ matrices are said to be row equivalent if one of them can be obtained from the other by using a number of elementary row operations.

Two systems of linear equations have the same set solutions if their augmented coefficient matrices are row equivalent.

A matrix is in echelon form if
1. rows with only 0's are below non-zero rows
2. the first non-zero element in a row is 1 (it is called the leading element or pivot)
3. a leading element in a row is in a column to the right of a leading element in row above it.

A matrix in echelon form is in reduced echelon form if
4. a column with a leading element (pivot) has 0 i all other rows.

Solution to a system of linear equations (when the augmented coefficient matrix is in reduced echelon form)

If the last column has a pivot then there is an equation of the form:

$$0x_0 + \ldots + 0x_{n-1} = 1,$$

and the system of equations has no solutions (it is inconsistent).

If there is a pivot in all columns except the last column then there is a unique solution to the system of equations.

If thre is no pivot in the last column and there is one more column with no pivot then there are infinitely many solutions.

MCG - 7

A: an $n \times n$ matrix.
Entry (i, j) is a_{ij}.

$\tilde{A}_{ij}$: an $(n-1) \times (n-1)$ matrix, obtained from A by deleting row i and column j.

Determinant.
$n = 1$: $\quad \det([a_{00}]) = a_{00}$

$n \geq 2$:

$$\det(A) = a_{00}\det(\tilde{A}_{00}) - a_{01}\det(\tilde{A}_{01}) + a_{02}\det(\tilde{A}_{02}) - \ldots + (-1)^{n-1}a_{0,n-1}\det(\tilde{A}_{0,n-1})$$

Expansion along row i:

$$\det(A) = \sum_{j=0}^{n-1} a_{ij}(-1)^{i+j} \det(\tilde{A}_{ij}).$$

Expansion along column j:

$$\det(A) = \sum_{i=0}^{n-1} a_{ij}(-1)^{i+j} \det(\tilde{A}_{ij}).$$

Properties of determinants:

$$\det(A^T) = \det(A), \qquad \det(AB) = \det(A)\det(B).$$

Elementary row operations on determinants.

Matrix B obtained from A by an elementary row operation:

1. multiply one of the rows by a scalar $k \neq 0$
   $\det(B) = k \det(A)$ i.e., $\det(A) = \frac{1}{k} \det(B)$.

2. replace row i by (row i) $+ k \cdot$ (row j), $i \neq j$
   the determinant is not changed: $\det(B) = \det(A)$.

3. swap two rows.
   the determinant changes sign: $\det(B) = -\det(A)$.

Inverse matrix.
An $n \times n$ matrix A has inverse matrix A^{-1} if

$$AA^{-1} = I, \quad A^{-1}A = I.$$

(If one of these equations is satisfied then they both are.)

A has an inverse if and only if det$(A) \neq 0$.

If application of row operations on $\begin{bmatrix} A & I \end{bmatrix}$ can lead to $\begin{bmatrix} I & B \end{bmatrix}$ then $A^{-1} = B$.

If $\begin{bmatrix} I & B \end{bmatrix}$ can not obtained from $\begin{bmatrix} A & I \end{bmatrix}$ by using row operations then A does not have an inverse.

If A and B are $n \times n$ matrices and both of them have an inverse then AB has an inverse:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Inverse of matrices of special type.

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}.$$

If a, b and c are non-zero then

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}.$$

Inverse of 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$.

An $n \times n$ matrix is said to be an orthogonal matrix if its column vectors are orthogonal and have length 1.

If A is an orthogonal matrix then $A^{-1} = A^T$.
Conversely, if $A^{-1} = A^T$ then A is an orthogonal matrix.

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An **affine transformation** $\mathcal{T} : \mathbb{R}^n \mapsto \mathbb{R}^m$ is a function satisfying

$$\mathcal{T}(a_0 P_0 + a_1 P_1) = a_0 \mathcal{T}(P_0) + a_1 \mathcal{T}(P_1),$$

for all points P_0, P_1 and all numbers a_0, a_1 where $a_0 + a_1 = 1$.

Let $\mathcal{T}$ be an affine transformation.
Let $\mathcal{S}(\mathbf{v}) = \mathcal{T}(O + \mathbf{v}) - \mathcal{T}(O)$, where $O = (0, \ldots, 0)$.
Then $\mathcal{S}$ is a linear transformation and therefore there exists a matrix A so that $\mathcal{S}(\mathbf{v}) = A\mathbf{v}$.

The columns of A are $\mathcal{S}(\mathbf{e}_0), \ldots, \mathcal{S}(\mathbf{e}_{n-1})$.

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$$\mathcal{T}(\mathbf{v}) = A\mathbf{v} + \mathbf{y},$$

where $\mathbf{y} = \mathcal{T}(O)$.

The affine transformation is represented by the following matrix

$$\begin{bmatrix} A & \mathbf{y} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

The inverse affine transformation $\mathcal{T}^{-1}$ is represented by the inverse matrix

$$\begin{bmatrix} A^{-1} & -A^{-1}\mathbf{y} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

The point P in $\mathbb{R}^n$ is represented by the following vector in $\mathbb{R}^{n+1}$

$$\begin{bmatrix} P \\ 1 \end{bmatrix}.$$

A **translation** by the vector $\mathbf{t}$ maps the point P to the point $P + \mathbf{t}$.

The matrix of this affine transformation is

$$\begin{bmatrix} I_n & \mathbf{t} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

Pure **rotation**. (pure = around axis through O). The rotation is then a linear transformation.

A linear transformation $T(\mathbf{v}) = A\mathbf{v}$ is a rotation
if and only if
A is an orthogonal matrix with $\det(A) = 1$.

A composition of two rotations is a rotation.

Rotation in $\mathbb{R}^3$ around the z-axis by the angle θ has matrix

$$R_z = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The affine matrix is

$$\begin{bmatrix} R_z & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

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Rotation in $\mathbb{R}^3$ by angle θ around axis with direction given by the vector $\mathbf{r}$.
If right hand thumb points in direction $\mathbf{r}$ then the fingers points in positive direction for θ.

Rotation by angle $-\theta$ around axis with vector $-\mathbf{r}$ is the same as rotation by angle θ around axis with vector $\mathbf{r}$.

Compute $\hat{\mathbf{r}} = \frac{1}{||\mathbf{r}||}\mathbf{r}$.

An arbitrary vector $\mathbf{v}$ is rotated in the vector $R(\mathbf{v})$, that can be computed using Rodrigues formula:

$$R(\mathbf{v}) = \cos(\theta)\mathbf{v} + (1 - \cos(\theta))(\mathbf{v}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + \sin(\theta)(\hat{\mathbf{r}} \times \mathbf{v}).$$

If $\hat{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then the matrix of the rotation is:

$$\mathbf{R}_{\hat{\mathbf{r}}\theta} = (1 - \cos(\theta) \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} + \cos(\theta) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin(\theta) \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}.$$

The matrix can also be written as

$$\mathbf{R}_{\hat{\mathbf{r}}\theta} = \begin{bmatrix} tx^2 + c & txy - sz & txz + sy \\ txy + sz & ty^2 + c & tyz - sx \\ txz - sy & tyz + sx & tz^2 + c \end{bmatrix},$$

where

$$c = \cos(\theta), \quad s = \sin(\theta), \quad t = 1 - \cos(\theta).$$

Rotation around the x-axis by angle θ_x [take $(x, y, z) = (1, 0, 0)$]:

$$\mathbf{R}_x = \mathbf{R}_{\mathbf{i}\theta_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & -\sin(\theta_x) \\ 0 & \sin(\theta_x) & \cos(\theta_x) \end{bmatrix}.$$

Rotation around the y-axis by angle θ_y [take $(x, y, z) = (0, 1, 0)$]:

$$\mathbf{R}_y = \mathbf{R}_{\mathbf{j}\theta_y} = \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta) \end{bmatrix}.$$

Rotation around the z-axis by angle θ_z [take $(x, y, z) = (0, 0, 1)$]:

$$\mathbf{R}_z = \mathbf{R}_{\mathbf{k}\theta_z} = \begin{bmatrix} \cos(\theta_z) & -\sin(\theta_z) & 0 \\ \sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix for rotation around the z-axis followed by rotation around the y-axis followed by rotation around the x-axis:

$$\mathbf{R}_x\mathbf{R}_y\mathbf{R}_z = \begin{bmatrix} CyCz & -CySz & Sy \\ SxSyCz + CxSz & -SxSySz + CxCz & -SxCy \\ -CxSyCz + SxSz & CxSySz + SxCz & CxCy \end{bmatrix},$$

where

$$Cx = \cos(\theta_x), \quad Sx = \sin(\theta_x),$$

$$Cy = \cos(\theta_y), \quad Sy = \sin(\theta_y),$$

$$Cz = \cos(\theta_z), \quad Sz = \sin(\theta_z).$$

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Reflection across a plane through $O = (0,0,0)$ with normal vector $\hat{\mathbf{n}}$, that has length 1.

The 3×3 matrix of the reflection:

$$\mathbf{I} - 2(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) = \begin{bmatrix} 1 - 2n_x^2 & -2n_x n_y & -2n_x n_z \\ -2n_x n_y & 1 - 2n_y^2 & -2n_y n_z \\ -2n_x n_z & -2n_y n_z & 1 - 2n_z^2 \end{bmatrix},$$

where $\hat{\mathbf{n}} = [n_x \; n_y \; n_z]^T$.
The 4×4 affine matrix is

$$\begin{bmatrix} \mathbf{I} - 2(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

Reflection across O has 3×3 matrix $-\mathbf{I}$.

Orthogonal matrices.

An orthogonal matrix has determinant 1 or -1.

En matrix A is an orthogonal matrix with determinant 1
if and only if
A is the matrix of a rotation.

The matrix of a reflection is an orthogonal matrix with determinant -1.
But only a small fraction of all orthogonal matrices with determinant -1 are matrices of a reflection.

Shear.

$\hat{\mathbf{n}}$: a vector with length 1.

$\mathbf{s}$: a vector orthogonal to $\hat{\mathbf{n}}$.

Shear plane: the plane through O with normal vector $\hat{\mathbf{n}}$.
Points on this plane are fixed.
An arbitrary vector $\mathbf{v}$ is mapped to $\mathbf{v} + (\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{s}$.
The 4×4 affine matrix for a shear is

$$H_{\hat{\mathbf{n}},\mathbf{s}} = \begin{bmatrix} \mathbf{I} + \mathbf{s} \otimes \hat{\mathbf{n}} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

$$\mathbf{s} \otimes \hat{\mathbf{n}} = \begin{bmatrix} s_x n_x & s_x n_y & s_x n_z \\ s_y n_x & s_y n_y & s_y n_z \\ s_z n_x & s_z n_y & s_z n_z \end{bmatrix},$$

where $\mathbf{s} = [s_x \ s_y \ s_z]^T$ and $\hat{\mathbf{n}} = [n_x \ n_y \ n_z]^T$.

Affine transformation around an arbitrary point.

$\mathbf{R}$ is the 3×3 matrix for a rotation around an axis through O or a shear or reflection around a plane through O.

The corresponding transformation around $C = O + \mathbf{x}$ has affine matrix

$$\begin{bmatrix} \mathbf{I} & \mathbf{x} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{x} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\mathbf{x} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

$\mathbf{R}$: 3×3 matrix for a rotation.

Compute **Euler angles** $\theta_x, \theta_y, \theta_z$ so that

$$\mathbf{R} = \mathbf{R}_x \mathbf{R}_y \mathbf{R}_z$$

where
$\mathbf{R}_x$ is rotation around the x-axis by angle θ_x
$\mathbf{R}_y$ is rotation around the y-axis by angle θ_y
$\mathbf{R}_z$ is rotation around the z-axis by angle θ_z.

The angle θ_y is determined by:

$$\sin\theta_y = \mathbf{R}_{02}, \quad \cos\theta_y = \sqrt{1 - \sin^2\theta_y}.$$

If $\cos\theta_y \neq 0$ then θ_x and θ_z are determined by

$$\sin\theta_x = -\frac{\mathbf{R}_{12}}{\cos\theta_y}, \quad \cos\theta_x = \frac{\mathbf{R}_{22}}{\cos\theta_y},$$

$$\sin\theta_z = -\frac{\mathbf{R}_{01}}{\cos\theta_y}, \quad \cos\theta_z = \frac{\mathbf{R}_{00}}{\cos\theta_y}.$$

If $\cos\theta_y = 0$ then choose $\theta_z = 0$ and θ_x is determined by

$$\sin\theta_x = \mathbf{R}_{21}, \quad \cos\theta_x = \mathbf{R}_{11}.$$

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$\mathbf{R}$ is a 3×3 rotation matrix.

Determine axis-angle representation of this rotation,

i.e., a vector $\hat{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and an angle θ so that $\mathbf{R} = \mathbf{R}_{\hat{\mathbf{r}}\theta}$.

Compute trace($\mathbf{R}$) $= R_{00} + R_{11} + R_{22}$.

Then $\theta = \cos^{-1}(\frac{\text{trace}(\mathbf{R})-1}{2})$. This gives $0^\circ \leq \theta \leq 180^\circ$.
$\cos^{-1}$ is also written as arccos.

If $\theta = 0°$: no rotation, $\hat{\mathbf{r}}$ is arbitrary (and $R = I$).

If $\theta \neq 0°$ and $\theta \neq 180°$:

$$\mathbf{r} = (R_{21} - R_{12}, R_{02} - R_{20}, R_{10} - R_{01}), \quad \hat{\mathbf{r}} = \frac{1}{||\mathbf{r}||}\mathbf{r}.$$

If $\theta = 180°$: Determine the largest of the numbers R_{00}, R_{11}, R_{22}.

R_{00} largest: $\quad x = \frac{1}{2}\sqrt{R_{00} - R_{11} - R_{22} + 1}, \quad y = \frac{R_{01}}{2x}, \quad z = \frac{R_{02}}{2x}.$

R_{11} largest: $\quad y = \frac{1}{2}\sqrt{R_{11} - R_{00} - R_{22} + 1}, \quad x = \frac{R_{01}}{2y}, \quad z = \frac{R_{12}}{2y}.$

R_{22} largest: $\quad z = \frac{1}{2}\sqrt{R_{22} - R_{00} - R_{11} + 1}, \quad x = \frac{R_{02}}{2z}, \quad y = \frac{R_{12}}{2z}.$

A **quaternion** q is written as

$$q = (w, x, y, z),$$

or

$$q = w + x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

If we let $\mathbf{v} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then we also write

$$q = (w, \mathbf{v}),$$

or

$$q = w + \mathbf{v}.$$

Addition of quaternions:

$(w_1, x_1, y_1, z_1) + (w_2, x_2, y_2, z_2) = (w_1 + w_2, x_1 + x_2, y_1 + y_2, z_1 + z_2)$.

Scalar multiplication:

$$a(w, x, y, z) = (aw, ax, ay, az).$$

Magnitude of a quaternion $q = (w, x, y, z)$:

$$||q|| = \sqrt{w^2 + x^2 + y^2 + z^2}.$$

If $q \neq (0, 0, 0, 0)$ then the quaternion

$$\frac{1}{||q||}q$$

has magnitude 1 and is said to be normalized.

Rotation around the axis $\hat{\mathbf{r}}$ with angle θ is represented by the quaternion

$$q = (\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}).$$

Or by

$$(\cos\left(\frac{360^\circ - \theta}{2}\right), \sin\left(\frac{360^\circ - \theta}{2}\right)(-\hat{\mathbf{r}})) = (-\cos\left(\frac{\theta}{2}\right), -\sin\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}) = -q.$$

The matrix for the rotation, represented by the normalized quaternion $q = (w, x, y, z)$:

$$\begin{bmatrix} 1 - 2y^2 - 2z^2 & 2xy - 2wz & 2xz + 2wy \\ 2xy + 2wz & 1 - 2x^2 - 2z^2 & 2yz - 2wx \\ 2xz - 2wy & 2yz + 2wx & 1 - 2x^2 - 2y^2 \end{bmatrix}$$

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Multiplication of quaternions:
When computing

$$(w_2 + x_2\mathbf{i} + y_2\mathbf{j} + z_2\mathbf{k})(w_1 + x_1\mathbf{i} + y_1\mathbf{j} + z_1\mathbf{k})$$

we may use the following:

$$\mathbf{ij} = \mathbf{k},\ \mathbf{ji} = -\mathbf{k}, \quad \mathbf{jk} = \mathbf{i},\ \mathbf{kj} = -\mathbf{i}, \quad \mathbf{ki} = \mathbf{j},\ \mathbf{ik} = -\mathbf{j}.$$

$$\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1.$$

We can also compute the product as

$$(w_2, \mathbf{v}_2)(w_1, \mathbf{v}_1) = (w_2 w_1 - \mathbf{v}_2 \cdot \mathbf{v}_1, w_1 \mathbf{v}_2 + w_2 \mathbf{v}_1 + \mathbf{v}_2 \times \mathbf{v}_1),$$

and in particular

$$(0, \mathbf{v}_2)(0, \mathbf{v}_1) = (-\mathbf{v}_2 \cdot \mathbf{v}_1, \mathbf{v}_2 \times \mathbf{v}_1).$$

All algebraic rules except the commutative law are valid. Usually:

$$q_1 q_2 \neq q_2 q_1.$$

Furthermore

$$||q_1 q_2|| = ||q_1|| \cdot ||q_2||.$$

Identity:

$$(w, \mathbf{v})(1, \mathbf{0}) = (1, \mathbf{0})(w, \mathbf{v}) = (w, \mathbf{v}).$$

Inverse: if $q = (w, \mathbf{v}) \neq (0, \mathbf{0})$ then q has inverse

$$q^{-1} = \frac{1}{||q||^2}(w, -\mathbf{v}).$$

If q is normalized ($||q|| = 1$) then

$$q^{-1} = (w, -\mathbf{v}).$$

The inverse quaternion satifies:

$$qq^{-1} = q^{-1}q = (1, \mathbf{0}).$$

Rotation by angle θ around the axis $\hat{\mathbf{r}}$ is represented by the quaternion

$$q = (\cos\left(\frac{\theta}{2}\right), \sin\left(\frac{\theta}{2}\right)\hat{\mathbf{r}}).$$

This quaternion satisfies $||q|| = 1$.

If $\mathbf{p}$ is a vector in 3D-space then let $R_q(\mathbf{p})$ be the vector that $\mathbf{p}$ is rotated into.

We think of $\mathbf{p}$ as a quaternion, $(0, \mathbf{p})$, and then we can compute $R_q(\mathbf{p})$ as follows

$$R_q(\mathbf{p}) = q\mathbf{p}q^{-1}.$$

If $q = (w, \mathbf{v})$ then this can also be computed as

$$R_q(\mathbf{p}) = (2w^2 - 1)\mathbf{p} + 2(\mathbf{v}\cdot\mathbf{p})\mathbf{v} + 2w(\mathbf{v} \times \mathbf{p}).$$

Converting from matrix representation of rotation to quaternion representation. (page 191)

R: a rotation matrix.

Compute:
trace$(R) = R_{00} + R_{11} + R_{22}$.
$\mathbf{r} = (R_{21} - R_{12}, R_{02} - R_{20}, R_{10} - R_{01})$.

$$q = (\text{trace}(R) + 1, \mathbf{r}) = \\ (R_{00} + R_{11} + R_{22} + 1, R_{21} - R_{12}, R_{02} - R_{20}, R_{10} - R_{01}).$$

Then the rotationen is represented by the normalized quaternion

$$\frac{1}{||q||}q.$$

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Converting from rotation matrix to normalized quaternion:

R: a 3×3 rotation matrix.

Compute:

$$q = (R_{00} + R_{11} + R_{22} + 1, R_{21} - R_{12}, R_{02} - R_{20}, R_{10} - R_{01}).$$

The rotation is then represented by the normalized quaternion

$$\frac{1}{||q||} q.$$

Alternative method (if trace(R) < 0):
Find the largest of the numbers R_{00}, R_{11}, R_{22}.

R_{00} largest: normalize the quaternionen

$$(R_{21} - R_{12}, R_{00} - R_{11} - R_{22} + 1, R_{01} + R_{10}, R_{02} + R_{20}).$$

R_{11} largest: normalize the quaternionen

$$(R_{02} - R_{20}, R_{01} + R_{10}, R_{11} - R_{00} - R_{22} + 1, R_{12} + R_{21}).$$

R_{22} largest: normalize the quaternionen

$$(R_{10} - R_{01}, R_{02} + R_{20}, R_{21} + R_{12}, R_{22} - R_{00} - R_{11} + 1).$$

If rotation around the axis $\mathbf{r}_1$ with angle θ_1 is represented by the quaternion q_1
and rotation around the axis $\mathbf{r}_2$ with angle θ_2 is represented by the quaternion q_2

then the composed rotation consisting of
    rotation around the axis $\mathbf{r}_1$ with angle θ_1
followed by
    rotation around the axis $\mathbf{r}_2$ with angle θ_2
is represented by the quaternion $q_2 q_1$.

Linear interpolation:

Find a parameterized line $Q(t)$, satisfying that $Q(t_i) = P_i$ and $Q(t_{i+1}) = P_{i+1}$, where P_i and P_{i+1} are points.

Solution

$$Q(t) = P_i + \frac{t - t_i}{t_{i+1} - t_i}(P_{i+1} - P_i),$$

when $t_i \leq t \leq t_{i+1}$.

Hermite curves:

Determine a curve $Q(t)$ satisfying that $Q(0) = P_0$, $Q(1) = P_1$, $\mathbf{Q}'(0) = \mathbf{P}'_0$ and $\mathbf{Q}'(1) = \mathbf{P}'_1$, where P_0 and P_1 are points and $\mathbf{P}'_0$ and $\mathbf{P}'_1$ are vectors.

Let $Q(t) = \mathbf{a}t^3 + \mathbf{b}t^2 + \mathbf{c}t + D$, where $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are vectors and D is a point.
Then $\mathbf{Q}'(t) = 3\mathbf{a}t^2 + 2\mathbf{b}t + \mathbf{c}$.

Requirement:

$$Q(0) = D = P_0, \quad Q(1) = \mathbf{a} + \mathbf{b} + \mathbf{c} + D = P_1.$$
$$\mathbf{Q}'(0) = \mathbf{c} = \mathbf{P}'_0, \quad \mathbf{Q}'(1) = 3\mathbf{a} + 2\mathbf{b} + \mathbf{c} = \mathbf{P}'_1.$$

Solution:

$\mathbf{a} = 2(P_0 - P_1) + \mathbf{P}'_0 + \mathbf{P}'_1$, $\mathbf{b} = 3(P_1 - P_0) - 2\mathbf{P}'_0 - \mathbf{P}'_1$,
$\mathbf{c} = \mathbf{P}'_0$ and $D = P_0$.

The Hermite curve satisfying the above condition can also be written as

$$Q(u) = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_0 \\ P_1 \\ \mathbf{P}'_0 \\ \mathbf{P}'_1 \end{bmatrix} = UMG,$$

where the 'vector' G is in fact a 4×3 matrix.

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Piecewise Hermite curves.

$P_0, P_1, \ldots, P_n$: points.

We want to find Hermite curves
$Q_0(u), Q_1(u), \ldots, Q_{n-1}(u)$,
so that each $Q_i(u)$ is a curve moving from P_i to P_{i+1} when u increases from 0 to 1:
$Q_i(0) = P_i$ and $Q_i(1) = P_{i+1}$ for all $i = 0, 1, \ldots, n-1$

In order to compute a unique Hermite $Q_0(u)$ we need to choose vectors $\mathbf{P}'_0$ and $\mathbf{P}'_1$ and require that $\mathbf{Q}'(0) = \mathbf{P}'_0$ and $\mathbf{Q}'(1) = \mathbf{P}'_1$ and similar for the other curves.

We may also use an automatic way to generate these conditions.

Automatic generation of Hermite curves.
A way to generate a piecewise Hermite curve through the points $P_0, P_1, \ldots, P_n$. The following conditions must be satisfied:

- $Q_i(1) = Q_{i+1}(0) = P_i$ for all $i = 0, 1, \ldots, n-2$ (previous slide)

- $\mathbf{Q}_i'(1) = \mathbf{Q}_{i+1}'(0)$ for all $i = 0, 1, \ldots, n-2$
  Q_{i+1} starts with the same velocity as Q_i has in the end.

- $\mathbf{Q}_i''(1) = \mathbf{Q}_{i+1}''(0)$ for alle $i = 0, 1, \ldots, n-2$
  Q_{i+1} starts with the same acceleration as Q_i has in the end.

- $\mathbf{Q}_0''(0) = \mathbf{0}$ og $\mathbf{Q}_{n-1}''(1) = \mathbf{0}$ (natural end conditions).
  No acceleration in the beginning and at the end.

In order to determine $\mathbf{P}'_0 = \mathbf{Q}'_0(0), \mathbf{P}'_1 = \mathbf{Q}'_1(0) = \mathbf{Q}'_0(1), \ldots, \mathbf{P}'_{n-1} = \mathbf{Q}'_{n-1}(0) = \mathbf{Q}'_{n-2}(1), \mathbf{P}'_n = \mathbf{Q}'_{n-1}(1)$ we derive the following system of equations from the equations on the previous slide (the matrix has size $(n+1) \times (n+1)$):

$$\begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 4 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 4 & 1 & \cdots & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & \cdots & 1 & 4 & 1 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 4 & 1 \\ 0 & 0 & \cdots & 0 & 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{P}'_0 \\ \mathbf{P}'_1 \\ \mathbf{P}'_2 \\ \vdots \\ \mathbf{P}'_{n-2} \\ \mathbf{P}'_{n-1} \\ \mathbf{P}'_n \end{bmatrix} = \begin{bmatrix} 3(P_1 - P_0) \\ 3(P_2 - P_0) \\ 3(P_3 - P_1) \\ \vdots \\ 3(P_{n-1} - P_{n-3}) \\ 3(P_n - P_{n-2}) \\ 3(P_n - P_{n-1}) \end{bmatrix}.$$

When $\mathbf{P}'_0, \mathbf{P}'_1, \ldots, \mathbf{P}'_n$ have been determined from the above equtions we can compute each Q_i as follows:

$$Q_i(u) = UMG,$$

where

$$U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} P_i \\ P_{i+1} \\ \mathbf{P}'_i \\ \mathbf{P}'_{i+1} \end{bmatrix}.$$

We have the following general formula:

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^{n-i} y^i, \quad \text{hvor} \quad \binom{n}{i} = \frac{n!}{i!(n-i)!},$$

and $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$.

In particular $(x+y)^2 = x^2 + 2xy + y^2$ and

$$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$

If we let $x = 1-u$ and $y = u$ we get

$$1 = (1-u)^3 + 3(1-u)^2u + 3(1-u)u^2 + u^3.$$

Bézier kurver:

$P_0, P_1, \ldots, P_n$ are points, called control points.

The Bézier curve is then

$$Q(u) = \sum_{i=0}^{n} \binom{n}{i} (1-u)^{n-i} u^i P_i.$$

It satisfies $Q(0) = P_0$ and $Q(1) = P_n$.

The most interesting case is $n = 3$:

$$Q(u) = (1-u)^3 P_0 + 3u(1-u)^2 P_1 + 3u^2(1-u) P_2 + u^3 P_3.$$

In the case $n = 3$ the Bézier curve can also be written as

$$Q(u) = J_{3,0}(u)P_0 + J_{3,1}(u)P_1 + J_{3,2}(u)P_2 + J_{3,3}(u)P_3,$$

where

$J_{3,0}(u) = (1-u)^3 = 1 - 3u + 3u^2 - u^3$
$J_{3,1}(u) = 3u(1-u)^2 = 3u - 6u^2 + 3u^3$
$J_{3,2}(u) = 3u^2(1-u) = 3u^2 - 3u^3$
$J_{3,3}(u) = u^3$

As $J_{3,0}(u) + J_{3,1}(u) + J_{3,2}(u) + J_{3,3}(u) = 1$ $Q(u)$ is an affine combination of P_0, P_1, P_2, P_3.

Furthermore $J_{3,0}(u) \geq 0$, $J_{3,1}(u) \geq 0$, $J_{3,2}(u) \geq 0$ and $J_{3,3}(u) \geq 0$. Thus $Q(u)$ is a convex combination of P_0, P_1, P_2, P_3 and the curve is contained in the convex hull of P_0, P_1, P_2, P_3.

The Bézier curve with $n = 3$ satifies $\mathbf{Q}'(0) = 3(P_1 - P_0)$ and $\mathbf{Q}'(1) = 3(P_3 - P_2)$ and the curve is the same as a Hermite curve

$$Q(u) = UMG,$$

where

$$U = \begin{bmatrix} u^3 & u^2 & u & 1 \end{bmatrix}, \quad M = \begin{bmatrix} 2 & -2 & 1 & 1 \\ -3 & 3 & -2 & -1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad G = \begin{bmatrix} P_0 \\ P_3 \\ 3(P_1 - P_0) \\ 3(P_3 - P_2) \end{bmatrix}.$$

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Interpolation of rotation.

p and q: rotation quaternions.

Spherical linear interpolation:

Determine the angle θ between p and q from $\cos\theta = p{\cdot}q$ (dot product of p and q).

Then the interpolation can be computed as follows:

$$\mathrm{slerp}(p, q, t) = \frac{\sin((1-t)\theta)p + \sin(t\theta)q}{\sin(\theta)}.$$

Linear interpolation:

Let

$$r = (1 - t)p + tq.$$

Then we get the linear interpolation by normalizing r:

$$\mathrm{lerp}(p, q, t) = \frac{1}{||r||} r.$$