MCG - 2

Operations on vectors:

Vectoraddition: if $\mathbf{v}$ and $\mathbf{w}$ are vectors then $\mathbf{v}+\mathbf{w}$ is a vector.

$$(v_0, v_1, \ldots, v_{n-1}) + (w_0, w_1, \ldots, w_{n-1}) = (v_0 + w_0, v_1 + w_1, \ldots, v_{n-1} + w_{n-1}).$$

Scalarmultiplication: if $\mathbf{v}$ is a vector and a is a number (scalar) then $a\mathbf{v}$ is a vector.

$$a(v_0, v_1, \ldots, v_{n-1}) = (av_0, av_1, \ldots, av_{n-1}).$$

Usual algebraic laws are valid for these operations.
E.g. $1\mathbf{v} = \mathbf{v}$ og $0\mathbf{v} = \mathbf{0}$.

If $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ are vectors and $a_0, a_1, \ldots, a_{n-1}$ are numbers then the expression

$$a_0\mathbf{v_0} + a_1\mathbf{v_1} + \ldots + a_{n-1}\mathbf{v_{n-1}}$$

is called a linear combination of $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$.

The set of vectors that are can be written as linear combinations of $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ is called the set (or subspace) spanned by $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$.

If one of the n vectors $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$ can be written as a linear combination of the other $n-1$ vectors then the vectors are said to be linearly dependent. Otherwise they are linearly independent.

The dotproduct of two vectors $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$ and $\mathbf{w} = (w_0, w_1, \ldots, w_{n-1})$ is defined by

$$\mathbf{v} \cdot \mathbf{w} = v_0 w_0 + v_1 w_1 + \ldots + v_{n-1} w_{n-1}.$$

The dotproduct also satisfies

$$\mathbf{v} \cdot \mathbf{w} = ||\mathbf{v}|| \; ||\mathbf{w}|| \cos\theta$$

where θ is the angle between the vectors.

$\mathbf{v}$ and $\mathbf{w}$ are orthogonal if $\mathbf{v} \cdot \mathbf{w} = 0$.

The length of $\mathbf{v}$ is $||\mathbf{v}|| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_0^2 + v_1^2 + \ldots + v_{n-1}^2}$.

The dotprodduct satisfies the following laws:

- $\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$
- $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$
- $a(\mathbf{v} \cdot \mathbf{w}) = (a\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (a\mathbf{w})$
- $\mathbf{v} \cdot \mathbf{v} \geq 0$ and
- $\mathbf{v} \cdot \mathbf{v} = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

The length of vectors satisfies:

- $||\mathbf{v}|| \geq 0 \quad$ and $\quad ||\mathbf{v}|| = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

- $||a\mathbf{v}|| = |a| \; ||\mathbf{v}||$

- $||\mathbf{v} + \mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.

These laws are also satisfied by the Manhattan norm

$$||\mathbf{v}||_{\ell_1} = |v_0| + |v_1| + \ldots + |v_{n-1}|$$

where $\mathbf{v} = (v_0, v_1, \ldots, v_{n-1})$.

Normalizing a vector $\mathbf{v} \neq \mathbf{0}$:

$$\hat{\mathbf{v}} = \frac{1}{||\mathbf{v}||}\mathbf{v}.$$

$\hat{\mathbf{v}}$ has the same direction as $\mathbf{v}$ and it has length 1.

The projection of a vector $\mathbf{v}$ on a vector $\mathbf{w} \neq \mathbf{0}$ er

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v} \cdot \mathbf{w}}{||\mathbf{w}||^2}\mathbf{w} = (\mathbf{v} \cdot \hat{\mathbf{w}})\hat{\mathbf{w}}.$$

The vector

$$\text{perp}_{\mathbf{w}}\mathbf{v} = \mathbf{v} - \text{proj}_{\mathbf{w}}\mathbf{v}$$

is orthogonal to $\mathbf{w}$.

A set of vectors $\{\mathbf{w_0}, \mathbf{w_1}, \ldots, \mathbf{w_{n-1}}\}$ is said to be orthonormal if the vectors are orthogonal and have length 1.

Gram-Schmidt orthogonalization of linearly independent vectors $\mathbf{v_0}, \mathbf{v_1}, \ldots, \mathbf{v_{n-1}}$:

- $\mathbf{w_0} = \mathbf{v_0}$
- $\mathbf{w_1} = \mathbf{v_1} - \mathrm{proj}_{\mathbf{w_0}}\mathbf{v_1}$
- $\mathbf{w_2} = \mathbf{v_2} - \mathrm{proj}_{\mathbf{w_0}}\mathbf{v_2} - \mathrm{proj}_{\mathbf{w_1}}\mathbf{v_2}$
- $\ldots$

In general:

$$\mathbf{w_i} = \mathbf{v_i} - \mathsf{proj}_{\mathbf{w_0}}\mathbf{v_i} - \ldots - \mathsf{proj}_{\mathbf{w_{i-1}}}\mathbf{v_i}.$$

Finally compute

$$\hat{\mathbf{w}}_\mathbf{0}, \hat{\mathbf{w}}_\mathbf{1}, \ldots, \hat{\mathbf{w}}_\mathbf{n-1}.$$

These vectors are orthonormal.