MCG - 7

A: an $n \times n$ matrix.
Entry (i, j) is a_{ij}.

$\tilde{A}_{ij}$: an $(n-1) \times (n-1)$ matrix, obtained from A by deleting row i and column j.

Determinant.
$n = 1$: $\det([a_{00}]) = a_{00}$

$n \geq 2$:

$$\det(A) = a_{00}\det(\tilde{A}_{00}) - a_{01}\det(\tilde{A}_{01}) + a_{02}\det(\tilde{A}_{02}) - \ldots + (-1)^{n-1}a_{0,n-1}\det(\tilde{A}_{0,n-1})$$

Expansion along row i:

$$\det(A) = \sum_{j=0}^{n-1} a_{ij}(-1)^{i+j} \det(\tilde{A}_{ij}).$$

Expansion along column j:

$$\det(A) = \sum_{i=0}^{n-1} a_{ij}(-1)^{i+j} \det(\tilde{A}_{ij}).$$

Properties of determinants:

$$\det(A^T) = \det(A), \qquad \det(AB) = \det(A)\det(B).$$

Elementary row operations on determinants.

Matrix B obtained from A by an elementary row operation:

1. multiply one of the rows by a scalar $k \neq 0$
   $\det(B) = k \det(A)$ i.e., $\det(A) = \frac{1}{k} \det(B)$.

2. replace row i by (row i) $+ k \cdot$ (row j), $i \neq j$
   the determinant is not changed: $\det(B) = \det(A)$.

3. swap two rows.
   the determinant changes sign: $\det(B) = -\det(A)$.

Inverse matrix.

An $n \times n$ matrix A has inverse matrix A^{-1} if

$$AA^{-1} = I, \quad A^{-1}A = I.$$

(If one of these equations is satisfied then they both are.)

A has an inverse if and only if $\det(A) \neq 0$.

If application of row operations on $\begin{bmatrix} A & I \end{bmatrix}$ can lead to $\begin{bmatrix} I & B \end{bmatrix}$ then $A^{-1} = B$.

If $\begin{bmatrix} I & B \end{bmatrix}$ can not obtained from $\begin{bmatrix} A & I \end{bmatrix}$ by using row operations then A does not have an inverse.

If A and B are $n \times n$ matrices and both of them have an inverse then AB has an inverse:

$$(AB)^{-1} = B^{-1}A^{-1}.$$

Inverse of matrices of special type.

$$\begin{bmatrix} 1 & 0 & x \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & -x \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix}.$$

If a, b and c are non-zero then

$$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}^{-1} = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}.$$

Inverse of 2×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

if $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \neq 0$.

An $n \times n$ matrix is said to be an orthogonal matrix if its column vectors are orthogonal and have length 1.

If A is an orthogonal matrix then $A^{-1} = A^T$.
Conversely, if $A^{-1} = A^T$ then A is an orthogonal matrix.