MCG - 9

Rotation in $\mathbb{R}^3$ by angle θ around axis with direction given by the vector $\mathbf{r}$.
If right hand thumb points in direction $\mathbf{r}$ then the fingers points in positive direction for θ.

Rotation by angle $-\theta$ around axis with vector $-\mathbf{r}$ is the same as rotation by angle θ around axis with vector $\mathbf{r}$.

Compute $\hat{\mathbf{r}} = \frac{1}{||\mathbf{r}||}\mathbf{r}$.

An arbitrary vector $\mathbf{v}$ is rotated in the vector $R(\mathbf{v})$, that can be computed using Rodrigues formula:

$$R(\mathbf{v}) = \cos(\theta)\mathbf{v} + (1 - \cos(\theta))(\mathbf{v}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} + \sin(\theta)(\hat{\mathbf{r}} \times \mathbf{v}).$$

If $\hat{\mathbf{r}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then the matrix of the rotation is:

$$\mathbf{R}_{\hat{\mathbf{r}}\theta} = (1-\cos(\theta)) \begin{bmatrix} x^2 & xy & xz \\ xy & y^2 & yz \\ xz & yz & z^2 \end{bmatrix} + \cos(\theta) \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \sin(\theta) \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}.$$

The matrix can also be written as

$$\mathbf{R}_{\hat{\mathbf{r}}\theta} = \begin{bmatrix} tx^2 + c & txy - sz & txz + sy \\ txy + sz & ty^2 + c & tyz - sx \\ txz - sy & tyz + sx & tz^2 + c \end{bmatrix},$$

where

$$c = \cos(\theta), \quad s = \sin(\theta), \quad t = 1 - \cos(\theta).$$

Rotation around the x-axis by angle θ_x [take $(x, y, z) = (1, 0, 0)$]:

$$\mathbf{R}_x = \mathbf{R}_{\mathbf{i}\theta_x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta_x) & -\sin(\theta_x) \\ 0 & \sin(\theta_x) & \cos(\theta_x) \end{bmatrix}.$$

Rotation around the y-axis by angle θ_y [take $(x, y, z) = (0, 1, 0)$]:

$$\mathbf{R}_y = \mathbf{R}_{\mathbf{j}\theta_y} = \begin{bmatrix} \cos(\theta_y) & 0 & \sin(\theta_y) \\ 0 & 1 & 0 \\ -\sin(\theta_y) & 0 & \cos(\theta) \end{bmatrix}.$$

Rotation around the z-axis by angle θ_z [take $(x, y, z) = (0, 0, 1)$]:

$$\mathbf{R}_z = \mathbf{R}_{\mathbf{k}\theta_z} = \begin{bmatrix} \cos(\theta_z) & -\sin(\theta_z) & 0 \\ \sin(\theta_z) & \cos(\theta_z) & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrix for rotation around the z-axis followed by rotation around the y-axis followed by rotation around the x-axis:

$$\mathbf{R}_x\mathbf{R}_y\mathbf{R}_z = \begin{bmatrix} CyCz & -CySz & Sy \\ SxSyCz + CxSz & -SxSySz + CxCz & -SxCy \\ -CxSyCz + SxSz & CxSySz + SxCz & CxCy \end{bmatrix},$$

where

$$Cx = \cos(\theta_x), \quad Sx = \sin(\theta_x),$$

$$Cy = \cos(\theta_y), \quad Sy = \sin(\theta_y),$$

$$Cz = \cos(\theta_z), \quad Sz = \sin(\theta_z).$$