MCG - 10

Reflection across a plane through $O = (0,0,0)$ with normal vector $\hat{\mathbf{n}}$, that has length 1.

The 3×3 matrix of the reflection:

$$\mathbf{I} - 2(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) = \begin{bmatrix} 1 - 2n_x^2 & -2n_x n_y & -2n_x n_z \\ -2n_x n_y & 1 - 2n_y^2 & -2n_y n_z \\ -2n_x n_z & -2n_y n_z & 1 - 2n_z^2 \end{bmatrix},$$

where $\hat{\mathbf{n}} = [n_x \ n_y \ n_z]^T$.
The 4×4 affine matrix is

$$\begin{bmatrix} \mathbf{I} - 2(\hat{\mathbf{n}} \otimes \hat{\mathbf{n}}) & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

Reflection across O has 3×3 matrix $-\mathbf{I}$.

Orthogonal matrices.

An orthogonal matrix has determinant 1 or -1.

En matrix A is an orthogonal matrix with determinant 1
if and only if
A is the matrix of a rotation.

The matrix of a reflection is an orthogonal matrix with determinant -1.
But only a small fraction of all orthogonal matrices with determinant -1 are matrices of a reflection.

Shear.

$\hat{\mathbf{n}}$: a vector with length 1.

$\mathbf{s}$: a vector orthogonal to $\hat{\mathbf{n}}$.

Shear plane: the plane through O with normal vector $\hat{\mathbf{n}}$.
Points on this plane are fixed.
An arbitrary vector $\mathbf{v}$ is mapped to $\mathbf{v} + (\hat{\mathbf{n}} \cdot \mathbf{v})\mathbf{s}$.
The 4×4 affine matrix for a shear is

$$H_{\hat{\mathbf{n}},\mathbf{s}} = \begin{bmatrix} \mathbf{I} + \mathbf{s} \otimes \hat{\mathbf{n}} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

$$\mathbf{s} \otimes \hat{\mathbf{n}} = \begin{bmatrix} s_x n_x & s_x n_y & s_x n_z \\ s_y n_x & s_y n_y & s_y n_z \\ s_z n_x & s_z n_y & s_z n_z \end{bmatrix},$$

where $\mathbf{s} = [s_x \ s_y \ s_z]^T$ and $\hat{\mathbf{n}} = [n_x \ n_y \ n_z]^T$.

Affine transformation around an arbitrary point.

$\mathbf{R}$ is the 3×3 matrix for a rotation around an axis through O or a shear or reflection around a plane through O.

The corresponding transformation around $C = O + \mathbf{x}$ has affine matrix

$$\begin{bmatrix} \mathbf{I} & \mathbf{x} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0}^T & 1 \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{x} \\ \mathbf{0}^T & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{R} & (\mathbf{I} - \mathbf{R})\mathbf{x} \\ \mathbf{0}^T & 1 \end{bmatrix}.$$

$\mathbf{R}$: 3×3 matrix for a rotation.

Compute **Euler angles** $\theta_x, \theta_y, \theta_z$ so that

$$\mathbf{R} = \mathbf{R}_x \mathbf{R}_y \mathbf{R}_z$$

where
$\mathbf{R}_x$ is rotation around the x-axis by angle θ_x
$\mathbf{R}_y$ is rotation around the y-axis by angle θ_y
$\mathbf{R}_z$ is rotation around the z-axis by angle θ_z.

The angle θ_y is determined by:

$$\sin\theta_y = \mathbf{R}_{02}, \quad \cos\theta_y = \sqrt{1 - \sin^2\theta_y}.$$

If $\cos\theta_y \neq 0$ then θ_x and θ_z are determined by

$$\sin\theta_x = -\frac{\mathbf{R}_{12}}{\cos\theta_y}, \quad \cos\theta_x = \frac{\mathbf{R}_{22}}{\cos\theta_y},$$

$$\sin\theta_z = -\frac{\mathbf{R}_{01}}{\cos\theta_y}, \quad \cos\theta_z = \frac{\mathbf{R}_{00}}{\cos\theta_y}.$$

If $\cos\theta_y = 0$ then choose $\theta_z = 0$ and θ_x is determined by

$$\sin\theta_x = \mathbf{R}_{21}, \quad \cos\theta_x = \mathbf{R}_{11}.$$