The complex logarithm

$e^z = e^w \iff e^{z-w} = 1 \iff z - w = 2\pi n i \quad , \quad n \in \mathbb{Z}.$

This shows that e^z is <u>not</u> one-to-one. Also, e^z is never zero.

For every $z \in \mathbb{C} \setminus \{0\}$ we define the logarithm of z to be the set:

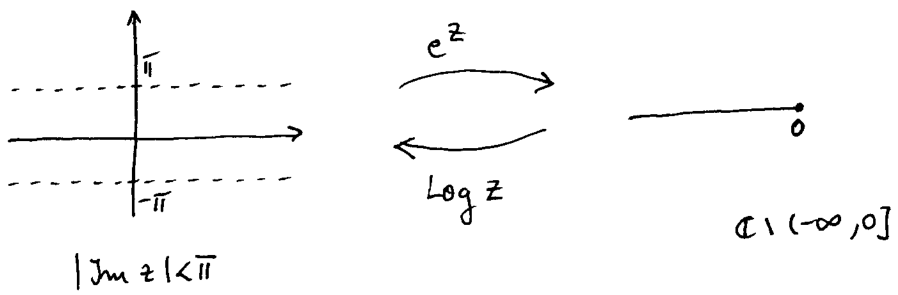
$$\log z = \{ w \in \mathbb{C} \mid e^w = z \}.$$

Every $z \in \mathbb{C}\setminus\{0\}$ can be written in polar coordinates as $z = re^{i\theta}$, with $r = |z|$ and $\theta \in [-\pi, \pi)$ uniquely determined.

For $z = re^{i\theta}$, $\quad e^w = re^{i\theta} \iff w = \log r + i\theta + 2\pi n i \quad , \quad n \in \mathbb{Z}.$

$$\log z = \log|z| + i\theta + 2\pi n i \quad , \quad n \in \mathbb{Z}.$$

If we restrict the exponential to the horizontal strip $|\mathrm{Im}\, z| < \pi$, it is one-to-one there and it maps the strip onto the slit plane $\mathbb{C} \setminus (-\infty, 0]$.

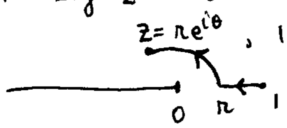


We define the <u>principal logarithm</u>, $\mathrm{Log}\, z$, to be the inverse of $e^z \big|_{|\mathrm{Im}\, z| < \pi}$.

$$\mathrm{Log}\, z = \log|z| + i\theta \qquad , \quad \theta \in \arg z \quad , \quad |\theta| < \pi$$

If $z = x$ is a real number, then $\mathrm{Log}\, z$ coincides with the real natural logarithm $\log x \ (= \ln x)$.

Notice that $\mathbb{C}\setminus(-\infty, 0]$ is starshaped and thus $\frac{1}{z}$ has a primitive there. We'll show that $\mathrm{Log}\, z$ is a primitive of $\frac{1}{z}$.



Consider γ_z to be the circuit formed by the line segment $L(1, r)$ and the arc of the circle of radius r, traced from r to $z = re^{i\theta}$.

$$\int_{\gamma_z} \frac{1}{z}\, dz = \int_1^r \frac{1}{t}\, dt + \int_0^\theta \frac{ire^{it}}{re^{it}}\, dt = \log r + i\theta = \mathrm{Log}\, z$$

Recall that $\log xy = \log x + \log y \quad , \quad \forall\, x, y > 0.$

This is <u>not true</u> in $\mathbb{C}$: $\quad \mathrm{Log}\, z_1 z_2 \neq \mathrm{Log}\, z_1 + \mathrm{Log}\, z_2.$