# Lecture Notes on Difference Equations

Arne Jensen
Department of Mathematical Sciences
Aalborg University, Fr. Bajers Vej 7G
DK-9220 Aalborg Ø, Denmark

Copyright ©2010 by Arne Jensen
All rights reserved

Version 1, October 10, 2010

# Contents

# 1 Introduction

These lecture notes are intended for the courses "Introduction to Mathematical Methods" and "Introduction to Mathematical Methods in Economics". They contain a number of results of a general nature, and in particular an introduction to selected parts of the theory of difference equations.

# 2 Notation and basic concepts

The positive integers $1, 2, 3, \ldots$ are denoted by $\mathbf{N}$. The non-negative integers are denoted by $\mathbf{N}_0$. All integers are denoted by $\mathbf{Z}$. The rational numbers are denoted by $\mathbf{Q}$. The real numbers are denoted by $\mathbf{R}$. We have the following obvious inclusions

$$\mathbf{N} \subset \mathbf{N}_0 \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}.$$

All inclusions are strict.

The main object of study in the theory of difference equations is sequences. A sequence of real numbers, indexed by either $\mathbf{Z}$ or $\mathbf{N}_0$, is written in either of two ways. It can be written as $x_n$ or as $x(n)$. The second notation makes it clear that a sequence is a function from either $\mathbf{Z}$ or $\mathbf{N}_0$ to $\mathbf{R}$. *We always use the notation $x(n)$ for a sequence.*

There is one property of the set $\mathbf{N}_0$ which is important. The set is *well-ordered*, which means that any non-empty subset of $\mathbf{N}_0$ contains a smallest element.

Sums play an important role in our presentation of the results on difference equations. Here are some concrete examples.

$$1 + 2 + 3 + 4 = \sum_{n=1}^{4} n = 10 \quad \text{and} \quad 2^2 + 3^2 + 4^2 + 5^2 = \sum_{n=2}^{5} n^2 = 54.$$

In general, the structure is

$$\sum_{n=n_{\text{first}}}^{n_{\text{last}}} x(n)$$

Here $n_{\text{first}}$ is called the lower limit and $n_{\text{last}}$ the upper limit. $x(n)$ is called the summand. It is a function of $n$, which we denote by $x(n)$. Sometimes we also write it as $x(n)$ the emphasize that it is a function.

Our results are sometimes expressed as indefinite sums. Here are two examples.

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \quad \text{and} \quad \sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}.$$

One important question is how to prove such general formulas. The technique used is called *proof by induction*. We will give a description of this technique. We have a certain statement, depending on an integer $n \in \mathbf{N}$. We would like to establish its validity for all $n \in \mathbf{N}$. The proof technique comprises two steps.

1. Basic step. Prove that the statement holds for $n = 1$.
2. Induction step. Prove that if the statement holds for $n$, then it also holds when $n$ is replaced by $n + 1$.

Verification of these two steps constitutes the proof of the statement for all integers $n \in \mathbf{N}$.

Let us illustrate the technique. We want to prove the formula

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \quad \text{for all } N \in \mathbf{N}.$$

For the first step we take $N = 1$. The formula then reads

$$1 = \frac{1(1+1)}{2},$$

which obviously is true. For the second step we assume that the formula is valid for some $N$ and consider the left hand side for $N + 1$.

$$\sum_{n=1}^{N+1} n = \Big(\sum_{n=1}^{N} n\Big) + (N+1) = \Big(\frac{N(N+1)}{2}\Big) + (N+1).$$

The second equality follows from our assumption. We now rewrite this last expression.

$$\frac{N(N+1)}{2} + N + 1 = \frac{N(N+1) + 2(N+1)}{2} = \frac{(N+1)(N+2)}{2}.$$

Thus we have shown that

$$\sum_{n=1}^{N+1} n = \frac{(N+1)((N+1)+1)}{2},$$

i.e. the formula holds with $N$ replaced by $N + 1$, and the proof is finished.

We also need a convenient notation for products. Here are two examples.

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = \prod_{n=1}^{5} n = 120 \quad \text{and} \quad 3 \cdot 5 \cdot 7 \cdot 9 = \prod_{n=1}^{4} (2n+1) = 945.$$

The terminology is analogous the the one used for sums. In particular, we will be using indefinite products. The product

$$\prod_{n=1}^{N} n$$

appears so often that is has a name. It is called the factorial of $N$, written as $N!$. So by definition

$$N! = \prod_{n=1}^{N} n.$$

It is a number that grows rapidly with $N$, as can be seen in these examples.

$$\begin{aligned} 10! &= 3628800, \\ 20! &= 2432902008176640000, \\ 30! &= 265252859812191058636308480000000. \end{aligned}$$

We have the convention that

$$0! = 1.$$

The general structure of a product is

$$\prod_{n=n_{\text{first}}}^{n_{\text{last}}} x(n).$$

**Important convention** We use the following conventions. If $n_1 > n_2$, then by definition

$$\sum_{n=n_1}^{n_2} a(n) = 0 \quad \text{and} \quad \prod_{n=n_1}^{n_2} a(n) = 1. \tag{2.1}$$

By this convention we have that

$$\sum_{n=0}^{-1} a(n) = 0 \quad \text{and} \quad \prod_{n=0}^{-1} a(n) = 1. \tag{2.2}$$

We now introduce the binomial formula. Given $x, y \in \mathbf{R}$, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{2.3}$$

Here the *binomial coefficients* are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad k = 0, \dots, n. \tag{2.4}$$

Recall our convention $0! = 1$. The binomial coefficients satisfy many identities. One of them is the following.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}, \quad k = 1, \dots, n. \tag{2.5}$$

This result is the consequence of the following computation.

$$\begin{aligned}\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!k}{k(k-1)!(n-k+1)!} + \frac{n!(n+1-k)}{k!(n-k)!(n+1-k)} \\ &= \frac{n!k + n!(n+1-k)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} \\ &= \binom{n+1}{k}.\end{aligned}$$

## Exercises

**Exercise 2.1.** Prove by induction that we have

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}.$$

**Exercise 2.2.** Let $q \in \mathbf{R}$ satisfy $q \neq 1$. Prove by induction that

$$\sum_{n=0}^{N} q^n = \frac{q^{N+1}-1}{q-1}. \tag{2.6}$$

What is $\sum_{n=0}^{N} q^n$ for $q = 1$?

**Exercise 2.3.** Prove by induction that we have

$$\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4}.$$

**Exercise 2.4.** Prove that

$$\sum_{n=1}^{N} n^3 = \Big(\sum_{n=1}^{N} n\Big)^2.$$

**Exercise 2.5.** Prove (2.3).

**Exercise 2.6.** Prove the following result

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

**Exercise 2.7.** Prove the following result

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

# 3 First order difference equations

In many cases it is of interest to model the evolution of some system over time. There are two distinct cases. One can think of time as a continuous variable, or one can think of time as a discrete variable. The first case often leads to differential equations. We will not discuss differential equations in these notes.

We consider a time period $T$ and observe (or measure) the system at times $t = nT$, $n \in \mathbf{N}_0$. The result is a sequence $x(0), x(1), x(2), \ldots$. In some cases these values are obtained from a function $f$, which is defined for all $t \geq$. In this case $x(n) = f(nT)$. This method of obtaining the values is called periodic sampling. One models the system using a difference equation, or what is sometimes called a recurrence relation.

In this section we will consider the simplest cases first. We start with the following equation

$$x(n+1) = ax(n), \quad n \in \mathbf{N}_0, \tag{3.1}$$

where $a$ is a given constant. The solution is given by

$$x(n) = a^n x(0). \tag{3.2}$$

The value $x(0)$ is called the *initial value.* To prove that (3.2) solves (3.1), we compute as follows.

$$x(n+1) = a^{n+1}x(0) = a(a^n x(0)) = ax(n).$$

**Example 3.1.** An amount of USD10,000 is deposited in a bank account with an annual interest rate of 4%. Determine the balance of the account after 15 years. This problem leads to the difference equation

$$b(n+1) = 1.04b(n), \quad b(0) = 10,000.$$

The solution is

$$b(n) = (1.04)^n 10,000,$$

in particular $b(15) = 18,009.44$.

We write the equation (3.1) as

$$x(n+1) - ax(n) = 0. \tag{3.3}$$

This equation is called a homogeneous first order difference equation with constant coefficients. The term homogeneous means that the right hand side is zero. A corresponding inhomogeneous equation is given as

$$x(n+1) - ax(n) = c, \tag{3.4}$$

where we take the right hand side to be a constant different from zero.

The equation (3.3) is called linear, since it satisfies the *superposition principle*. Let $y(n)$ and $z(n)$ be two solutions to (3.3), and let $\alpha, \beta \in \mathbf{R}$ be two real numbers. Define $w(n) = \alpha y(n) + \beta z(n)$. Then $w(n)$ also satisfies (3.3), as the following computation shows.

$$\begin{aligned} w(n+1) - aw(n) &= \alpha y(n+1) + \beta z(n+1) - a(\alpha y(n) + \beta z(n)) \\ &= \alpha(y(n+1) - ay(n)) + \beta(z(n+1) - az(n)) = \alpha 0 + \beta 0 = 0. \end{aligned}$$

We now solve (3.4). The idea is to compute a number of terms, guess the structure of the solution, and then prove that we have indeed found the solution. First we compute a number of terms. In the computation of $x(2)$ we give all intermediate steps. These are omitted in the computation of $x(3)$ etc.

$$\begin{aligned} x(1) &= ax(0) + c, \\ x(2) &= ax(1) + c = a(ax(0) + c) + c = a_2 x(0) + ac + c, \\ x(3) &= ax(2) + c = a^3 x(0) + a_2 c + ac + c, \\ x(4) &= ax(3) + c = a^4 x(0) + a^3 c + a_2 c + ac + c, \\ x(5) &= ax(4) + c = a^5 x(0) + a^4 c + a^3 c + a_2 c + ac + c, \\ &\;\;\vdots \\ x(n) &= a^n x(0) + c \sum_{k=0}^{n-1} a^k. \end{aligned}$$

Thus we have guessed that the solution is given by

$$x(n) = a^n x(0) + c \sum_{k=0}^{n-1} a^k. \tag{3.5}$$

To prove that (3.5) is a solution to (3.4), we must prove (3.5) satisfies this equation. We compute as follows.

$$\begin{aligned} x(n+1) &= a^{n+1} x(0) + c \sum_{k=0}^{n} a^k \\ &= a^{n+1} x(0) + c(1 + a + a_2 + \cdots + a^{n-1} + a^n) \\ &= a(a^n x(0)) + c + a(c(1 + a + a_2 + \cdots + a^{n-1})) \\ &= a\Big(a^n x(0) + c \sum_{k=0}^{n-1} a^k\Big) + c \end{aligned}$$

$$= ax(n) + c.$$

Thus we have shown that (3.5) is a solution to (3.4). For $a \neq 1$ the solution (3.5) can be rewritten using the result (2.6):

$$x(n) = a^n x(0) + c\frac{a^n - 1}{a - 1}. \tag{3.6}$$

In the general case both $a$ and $c$ will be functions of $n$. We have the following result.

**Theorem 3.2.** *Let $a(n)$, and $c(n)$, $n \in \mathbf{N}_0$, be real sequences. Then the linear first order difference equation*

$$x(n+1) = a(n)x(n) + c(n) \quad \textit{with initial condition } x(0) = y_0 \tag{3.7}$$

*has the solution*

$$y(n) = \prod_{k=0}^{n-1} a(k) y_0 + \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n-1} a(j) \right) c(k). \tag{3.8}$$

*The solution is unique.*

*Proof.* We define the sequence $y(n)$ by (3.8). We must show that it satisfies the equation (3.7) and the initial condition. Due to the convention (2.1) the initial condition is trivially satisfied. We first write out the expression for $y(n+1)$

$$y(n+1) = \prod_{k=0}^{n} a(k) y_0 + \sum_{k=0}^{n} \left( \prod_{j=k+1}^{n} a(j) \right) c(k).$$

We then rewrite the last term above as follows, using (2.1).

$$\begin{aligned}
\sum_{k=0}^{n} \left( \prod_{j=k+1}^{n} a(j) \right) c(k) &= \prod_{j=n+1}^{n} a(j) c(n) + \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n} a(j) \right) c(k) \\
&= c(n) + \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n} a(j) \right) c(k) = c(n) + a(n) \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n-1} a(j) \right) c(k).
\end{aligned}$$

Using this result we get

$$y(n+1) = a(n) \prod_{k=0}^{n-1} a(k) y(0) + c(n) + a(n) \sum_{k=0}^{n-1} \left( \prod_{j=k+1}^{n-1} a(j) \right) c(k),$$

which implies

$$y(n+1) = a(n)y(n) + c(n).$$

Thus we have shown that $y(n)$ is a solution. Finally we must prove uniqueness. Assume that we have two solutions $y(n)$ and $\tilde{y}(n)$, which satisfy (3.7), i.e. both the equation and the initial condition are satisfied by both solutions. Now consider $\{n \in \mathbf{N}_0 \mid y(n) \neq \tilde{y}(n)\}$. Let $n_0$ be the smallest integer in this set. Assume $n_0 \geq 1$. By the definition of $n_0$ we have $y(n_0 - 1) = \tilde{y}(n_0 - 1)$, and then

$$y(n_0) = a(n_0 - 1)y(n_0 - 1) + c(n_0 - 1) = a(n_0 - 1)\tilde{y}(n_0 - 1) + c(n_0 - 1) = \tilde{y}(n_0),$$

which is a contradiction. Thus we must have $n_0 = 0$. But $y(0) = \tilde{y}(0)$, since the two equations satisfy the same initial condition. It follows that the solution is unique. □

**Example 3.3.** Let us consider the payment of a loan. Payments are made periodically, e.g. once a month. The interest rate per period is $100r\%$. The payment at the end of each period is denoted $p(n)$. The initial loan is $q(0)$. The outstanding balance after $n$ payments is denoted $q(n)$. Thus $q(n)$ must satisfy the difference equation

$$q(n+1) = (1+r)q(n) - p(n). \tag{3.9}$$

The solution follows from (3.8).

$$q(n) = (1+r)^n q(0) - \sum_{k=0}^{n-1} (1+r)^{n-k-1} p(n). \tag{3.10}$$

Often the loan is paid back in equal installments, i.e. $p(n) = p$ for all $n$. Then the above sum can be computed. We get the result

$$q(n) = (1+r)^n q(0) - ((1+r)^n - 1)\frac{p}{r}. \tag{3.11}$$

Suppose that we want to pay back the loan in $N$ installments. Then the installment is determined by

$$p = q(0)\frac{r}{1-(1+r)^{-N}} \tag{3.12}$$

## Exercises

**Exercise 3.1.** Fill in the details in Example 3.3. In particular the computations leading to (3.11).

**Exercise 3.2.** Discuss the applications of the results in Example 3.3.

**Exercise 3.3.** Adapt the results in the Example 3.3 to the case, where initially no installments are paid.

**Exercise 3.4.** Discuss the application to loans with a variable interest rate of the results in this section.

**Exercise 3.5.** Implement the various formulas for interest computation and loan amortization on a programmable calculator or in Maple. In particular, implement the formulas for loans with a variable interest rate and try them out on some real world examples.

# 4 Difference calculus

Before we proceed to the study of general difference equations, we establish some results on the difference calculus. We denote all functions from $\mathbf{Z}$ to $\mathbf{R}$ by $\mathsf{S}(\mathbf{Z})$, and all functions from $\mathbf{N}_0$ to $\mathbf{R}$ by $\mathsf{S}(\mathbf{N}_0)$.

The set $\mathsf{S}(\mathbf{Z})$ is a real vector space. See [1] for the definition.

**Proposition 4.1.** *The set* $\mathsf{S}(\mathbf{Z})$ *is a real vector space, if the addition is defined as*

$$(x+y)(n) = x(n) + y(n), \quad x, y \in \mathsf{S}(\mathbf{Z}),$$

*and the scalar multiplication as*

$$(ax)(n) = ax(n), \quad a \in \mathbf{R},\ x \in \mathsf{S}(\mathbf{Z}).$$

Below we give definitions and results for $x \in \mathsf{S}(\mathbf{Z})$. To apply these results to functions (sequences) on $\mathbf{N}_0$, we consider $\mathsf{S}(\mathbf{N}_0)$ as a subset of $\mathsf{S}(\mathbf{Z})$. This is done in the following manner. Given $x \in \mathsf{S}(\mathbf{N}_0)$, we define

$$(\iota x)(n) = \begin{cases} x(n) & \text{for } n \geq 0, \\ 0 & \text{for } n < 0. \end{cases}$$

A function that maps a function $x(n)$ to a new function $y(n)$ is called an *operator*. An example is the operator $\iota \colon \mathsf{S}(\mathbf{N}_0) \to \mathsf{S}(\mathbf{Z})$ defined above. We define the operators $\Delta$, $S$, and $I$ as follows:

**Definition 4.2.** The *shift operator* $S \colon \mathsf{S}(\mathbf{Z}) \to \mathsf{S}(\mathbf{Z})$ is defined by

$$(Sx)(n) = x(n+1). \tag{4.1}$$

The *difference operator* $\Delta$ is defined by

$$(\Delta x)(n) = x(n+1) - x(n). \tag{4.2}$$

The *identity operator* $I$ is defined by

$$(Ix)(n) = x(n). \tag{4.3}$$

The relation between the three operators is

$$\Delta = S - I. \tag{4.4}$$

The operators $S$ and $\Delta$ are linear. We recall from [1] that an operator $U \colon \mathsf{S}(\mathbf{Z}) \to \mathsf{S}(\mathbf{Z})$ is said to be linear, if it satisfies

$$U(x+y) = Ux + Uy \quad \text{for all } x, y \in \mathsf{S}(\mathbf{Z}), \tag{4.5}$$

$$U(ax) = aUx \quad \text{for all } x \in \mathsf{S}(\mathbf{Z}) \text{ and } a \in \mathbf{R}. \tag{4.6}$$

We recall that composition of two linear operators $U, V \colon \mathsf{S}(\mathbf{Z}) \to \mathsf{S}(\mathbf{Z})$ is defined as $(U \circ Vx)(n) = (U(Vx))(n)$. If $U = V$, we write $U \circ U = U_2$. Usually we also write $UV$ instead of $U \circ V$.

# 5 Second order linear difference equations

We will now present the theory of second order linear difference equations. In contrast to the first order case, there is no general formula that gives the solution to all such equations. One has to impose additional conditions in order to get a general formula.

The general form of a second order linear difference equation is

$$x(n+2) + b(n)x(n+1) + c(n)x(n) = f(n), \quad n \in \mathbf{N}_0. \tag{5.1}$$

Here $b(n)$, $c(n)$, $f(n)$ are given sequences. If $f(n) = 0$ for all $n$, then the equation is homogeneous, viz.

$$x(n+2) + b(n)x(n+1) + c(n)x(n) = 0, \quad n \in \mathbf{N}_0. \tag{5.2}$$

If we define the operator

$$(Lx)(n) = x(n+2) + b(n)x(n+1) + c(n)x(n),$$

then $L\colon \mathsf{S}(\mathbf{N}_0) \to \mathsf{S}(\mathbf{N}_0)$ is a linear operator, see Section 4.

We need some techniques and results from linear algebra in order to discuss the second and higher order equations.

**Definition 5.1.** Let $x_j \in \mathsf{S}(\mathbf{N}_0)$, $j = 1, \dots, N$. The list of vectors $x_1, x_2, \dots, x_N$ is said to be *linearly independent*, if for all $c_1, c_2, \dots, c_N$

$$c_1x_1 + c_2x_2 + \cdots + c_Nx_N = 0 \quad \text{implies} \quad c_1 = 0, c_2 = 0, \dots, c_N = 0. \tag{5.3}$$

If the list of vectors is not linearly independent, it is said to be *linearly dependent*.

**Remark 5.2.** We make a number of remarks on this definition.

(i) The definition is the same as in [1], and many of the results stated there carry over to the present more abstract framework.

(ii) We call the collection of vectors $x_1, x_2, \dots, x_N$ a list, since the elements are viewed as ordered. In particular, in contrast to a set, repetition of entries is significant.

(iii) Let us state explicitly what it means that the list of vectors $x_1, x_2, \dots, x_N$ is *linearly dependent*. It means that there exist $c_1, c_2, \dots, c_N$ with at least one $c_j \neq 0$, such that

$$c_1x_1(n) + c_2x_2(n) + \cdots + c_Nx_N(n) = 0 \quad \text{for all } n \in \mathbf{N}_0. \tag{5.4}$$

We will need some results to prove linear independence of vectors in $\mathsf{S}(\mathbf{N}_0)$. We give the general definition here. In this section we use it only for $N = 2$.

**Definition 5.3.** Let $N \geq 2$. Let $x_1, x_2, \dots, x_N \in \mathsf{S}(\mathbf{N}_0)$. Then we define the *Casoratian* by

$$W(n) = \det \begin{bmatrix} x_1(n) & x_2(n) & \cdots & x_N(n) \\ x_1(n+1) & x_2(n+1) & \cdots & x_N(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(n+N-1) & x_2(n+N-1) & \cdots & x_N(n+N-1) \end{bmatrix} \tag{5.5}$$

Note that the Casoratian is a function of $n$. It also depends on the vectors $x_1, x_2, \dots, x_N$, but this is not made explicit in the notation.

The Casoratian gives us a convenient method to determine, whether a given set of vectors is linearly independent.

**Proposition 5.4.** *Let $N \geq 2$. Let $x_1, x_2, \dots, x_N \in \mathsf{S}(\mathbf{N}_0)$. If there exists an $n_0 \in \mathbf{N}_0$, such that $W(n_0) \neq 0$, then $x_1, x_2, \dots, x_N$ are linearly independent.*

*Proof.* We give the proof in the case $N = 2$. Thus we have sequences $x_1$, $x_2$, and $n_0 \in \mathbf{N}_0$, such that

$$W(n_0) = \det \begin{bmatrix} x_1(n_0) & x_2(n_0) \\ x_1(n_0+1) & x_2(n_0+1) \end{bmatrix} \neq 0. \tag{5.6}$$

Now assume that we have a linear combination

$$c_1x_1 + c_2x_2 = 0.$$

More explicitly, this means that $c_1x_1(n) + c_2x_2(n) = 0$ for all $n \in \mathbf{N}_0$. In particular, we have

$$\begin{aligned} c_1x_1(n_0) + c_2x_2(n_0) &= 0,\\ c_1x_1(n_0+1) + c_2x_2(n_0+1) &= 0. \end{aligned}$$

But then $c_1 = c_2 = 0$, by well-known results from linear algebra, see [1].

The general case is left as an exercise. □

**Lemma 5.5.** *Assume that $x_1$ and $x_2$ are two solution to the homogeneous equation* (5.2)*. Let $W(n)$ be the Casoration of these solutions, given by* (5.5)*, $N = 2$. Then we have for $n_0 \in \mathbf{N}_0$ that for all $n \geq n_0$*

$$W(n) = W(n_0)\prod_{k=n_0}^{n-1} c(n). \tag{5.7}$$

*Proof.* The equation (5.2) implies

$$x_j(n+2) = -c(n)x_j(n) - b(n)x_j(n+1).$$

Then we have

$$\begin{aligned} W(n+1) &= \det\begin{bmatrix} x_1(n+1) & x_2(n+1)\\ x_1(n+2) & x_2(n+2)\end{bmatrix}\\ &= \det\begin{bmatrix} x_1(n+1) & x_2(n+1)\\ -c(n)x_1(n) - b(n)x_1(n+1) & -c(n)x_2(n) - b(n)x_2(n+1)\end{bmatrix}\\ &= \det\begin{bmatrix} x_1(n+1) & x_2(n+1)\\ -c(n)x_1(n) & -c(n)x_2(n)\end{bmatrix} = c(n)W(n). \end{aligned}$$

Solving the linear first order difference equation $W(n+1) = c(n)W(n)$ with initial value $W(n_0)$ (see Theorem 3.2), we conclude the proof. □

## 5.1 The constant coefficient case: Homogeneous equation

In this case the functions $b(n)$ and $c(n)$ are constants, denoted by $b$ and $c$. We start by solving the homogeneous equation. Thus we consider the equation

$$x(n+2) + bx(n+1) + cx(n) = 0, \quad n \in \mathbf{N}_0, \quad \text{with } b, c \in \mathbf{R}. \tag{5.8}$$

We now go through the steps leading to the complete solution to this equation, and then at the end we summarize the results in a theorem.

We assume that $c \neq 0$, since otherwise the equation is a first order equation for the function $y(n) = x(n+1)$, which we have already solved. To solve the equation (5.8) we try to find solutions of the form $x(n) = r^n$, where $r \neq 0$, and $r$ may be either real or complex. We will see below why we have to allow complex solutions. Insert $x(n) = r^n$ into (5.8) and use $r \neq 0$ to get the equation

$$r^2 + br + c = 0. \tag{5.9}$$

This equation is called the *characteristic equation* of (5.8).

There are now three possibilities.

**Case 1** If $b^2 - 4c > 0$, then (5.9) has two different real roots, which we denote by $r_1$ and $r_2$.

**Case 2** If $b^2 - 4c = 0$, then (5.9) has a real double root, which we denote by $r_0$.

**Case 3** If $b^2 - 4c < 0$, then (5.9) has pair of complex conjugate roots, which we denote by $r_\pm = \alpha \pm \beta$, $\beta > 0$.

Consider first **Case 1**. Let $x_1(n) = r_1^n$ and $x_2(n) = r_2^n$, $n \in \mathbf{N}_0$. We now use Proposition 5.4 with $n_0 = 0$. We have

$$W(0) = \det \begin{bmatrix} r_1^0 & r_2^0 \\ r_1^1 & r_2^1 \end{bmatrix} = r_2 - r_1 \neq 0.$$

Thus we have found two linearly independent solutions to (5.8). Note that the solutions are real.

Next we consider **Case 3**. Since we assume that the coefficients in (5.8) are real, we would like to find real solutions. We state the following result.

**Proposition 5.6.** *Let $y$ be a complex solution to* (5.8)*. Then $x_1(n) = \operatorname{Re} y(n)$ and $x_2(n) = \operatorname{Im} y(n)$ are real solutions to* (5.8)*.*

*Proof.* By assumption we have that

$$y(n+2) + by(n+1) + cy(n) = 0 \quad \text{for all } n \in \mathbf{N}_0.$$

Taking the real part and using that $b, c$ are real, we get

$$\operatorname{Re} y(n+2) + b \operatorname{Re} y(n+1) + c \operatorname{Re} y(n) = 0 \quad \text{for all } n \in \mathbf{N}_0,$$

which proves the result for $x_1$. The proof for $x_2$ follows in the same manner by taking imaginary parts. □

We now use some results concerning complex numbers, see [1, Appendix C]. We know that $y(n) = r_+^n$ is a solution, and we use Proposition 5.6 to find two real solutions, given by $x_1(n) = \operatorname{Re} r_+^n$ and $x_1(n) = \operatorname{Im} r_+^n$. We now rewrite these two solutions. Let

$$\rho = |r_+| = \sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad \theta = \operatorname{Arg} r_+. \tag{5.10}$$

We recall that we have $0 < \theta < \pi$, since we have $\beta > 0$. Now $r_+ = \rho e^{i\theta}$ and then $r_+^n = \rho^n e^{in\theta}$. Taking real and imaginary parts and using the de Moivre formula, we get

$$x_1(n) = \rho^n \cos(n\theta) \quad \text{and} \quad x_2(n) = \rho^n \sin(n\theta). \tag{5.11}$$

We use Proposition 5.4 to verify that $x_1$ and $x_2$ are linearly independent. We have

$$W(0) = \det \begin{bmatrix} 1 & 0 \\ \rho\cos(\theta) & \rho\sin(\theta) \end{bmatrix} = \rho \sin(\theta) \neq 0,$$

since $\rho > 0$ and $0 < \theta < \pi$.

It remains to consider **Case 2**. We have one real solution $x_1$ given by $x_1(n) = r_0^n$. We note that $r_0 = -\frac{b}{2}$. We need to find another solution. To do this we use a general procedure known as *reduction of order*. We try to find the second solution in the form $y(n) = u(n)x_1(n)$. Using the notation $(\Delta u)(n) = u(n+1) - u(n)$, see (4.2), we have

$$y(n+1) = u(n)x_1(n+1) + (\Delta u)(n)x(n+1), \tag{5.12}$$

$$y(n+2) = u(n)x_1(n+2) + (\Delta u)(n)x_1(n+2) + (\Delta u)(n+1)x_1(n+2). \tag{5.13}$$

We now compute as follows, using $x_1(n+2) + bx_1(n+1) + cx_1(n) = 0$,

$$\begin{aligned} &y(n+2) + by(n+1) + cy(n) \\ &\qquad = u(n)x_1(n+2) + (\Delta u)(n)x_1(n+2) + (\Delta u)(n+1)x_1(n+2) \\ &\qquad\quad + b\big(u(n)x_1(n+1) + (\Delta u)(n)x_1(n+1)\big) \\ &\qquad\quad + c\big(u(n)x_1(n)\big) \\ &\qquad = (\Delta u)(n+1)x_1(n+2) + (\Delta u)(n)\big(x_1(n+2) + bx_1(n+1)\big). \end{aligned} \tag{5.14}$$

Now we look for $y(n)$ satisfying $y(n+2) + by(n+1) + cy(n) = 0$. Using $x_1(n) = r_0^n$, we get from (5.14) after division by $x_1(n+2)$ the equation

$$(\Delta u)(n+1) + (\Delta u)(n)\big(1 + b\frac{x_1(n+1)}{x_1(n+2)}\big) = (\Delta u)(n+1) + (\Delta u)(n)\big(1 + b\frac{1}{r_0}\big) = 0.$$

We have

$$1 + b\frac{1}{r_0} = 1 + b\frac{1}{-\frac{b}{2}} = -1.$$

Solving the first order difference equation $(\Delta u)(n+1) - (\Delta u)(n) = 0$, we get $(\Delta u)(n) = c_1$, and then solving the first order equation $u(n+1) - u(n) = c_1$, we get

$$u(n) = c_1 n + c_2, \quad c_1, c_2 \in \mathbf{R}.$$

Thus we have found the solutions $y(n) = (c_1 n + c_2)r_0^n$. $c_1 = 0$ leads to the already known solutions $c_2 r_0^n$, so we take $c_2 = 0$ and $c_1 = 1$ to get the solution $x_2(n) = nr_0^n$. We compute the Casoration at zero of the two solutions that we have found.

$$W(0) = \det\begin{bmatrix} 1 & 0 \\ r_0 & 1r_0 \end{bmatrix} = r_0 \neq 0.$$

Thus we have found two linearly independent solutions.

We summarize the above results in the following Theorem.

**Theorem 5.7.** *The second order homogeneous difference equation with constant real coefficients*

$$x(n+2) + bx(n+1) + cx(n) = 0, \quad b, c \in \mathbf{R},\ c \neq 0, \quad n \in \mathbf{N}_0, \tag{5.15}$$

*always has two real linearly independent solutions $x_1$ and $x_2$. They are determined from the* characteristic equation

$$r^2 + br + c = 0. \tag{5.16}$$

(i) *If $b^2 - 4c > 0$, the two real solutions to* (5.16) *are denoted by $r_1$ and $r_2$. The two linearly independent solutions to* (5.15) *are given by*

$$x_1(n) = r_1^n \quad \textit{and} \quad x_2(n) = r_2^n, \quad n \in \mathbf{N}_0. \tag{5.17}$$

(ii) *If $b^2 - 4c = 0$, the real solution to* (5.16) *is denoted by $r_0$. The two linearly independent solutions to* (5.15) *are given by*

$$x_1(n) = r_0^n \quad \textit{and} \quad x_2(n) = nr_0^n, \quad n \in \mathbf{N}_0. \tag{5.18}$$

(iii) *If* $b^2 - 4c < 0$*, the two complex conjugate solution to* (5.16) *are denoted by* $r_\pm = \alpha \pm \beta$*,* $\beta > 0$*. Let* $r_+ = \rho e^{i\theta}$*,* $\rho = |r_+|$*,* $\theta = \operatorname{Arg} r_+$*. The two linearly independent solutions to* (5.15) *are given by*

$$x_1(n) = \rho^n \cos(n\theta) \quad \textit{and} \quad x_2(n) = \rho^n \sin(n\theta), \quad n \in \mathbf{N}_0. \tag{5.19}$$

Next we show how to describe all solutions to the equation (5.15).

**Theorem 5.8** (Superposition principle)**.** *Let* $x_1$ *and* $x_2$ *be solutions to* (5.15)*. Let* $c_1, c_2 \in \mathbf{R}$*. Then* $y = c_1x_1 + c_2x_2$ *is a solution to* (5.15)*.*

*Proof.* The proof is left as an exercise. □

**Theorem 5.9** (Uniqueness)**.** *A solution* $y$ *to* (5.15) *is uniquely determined by the initial values* $y0 = y(0)$ *and* $y1 = y(1)$*.*

*Proof.* Assume that we have two solutions $y_1$ and $y_2$ to (5.15), with the initial values $y0$ and $y1$, i.e. $y_1(0) = y_2(0) = y0$ and $y_1(1) = y_2(1) = y1$. We must show that $y_1(n) = y_2(n)$ for all $n \in \mathbf{N}_0$. Let $y(n) = y_1(n) - y_2(n)$. Then by Theorem 5.8 $y$ satisfies (5.15) with initial values zero. It follows from (5.15), written as

$$x(n+2) = -bx(n+1) - cx(n),$$

that $y(n) = 0$ for all $n \in \mathbf{N}_0$. More precisely, one proves this by induction. □

Before proving the next Theorem we need the following result, which complements Proposition 5.4.

**Lemma 5.10.** *Assume that* $x_1$ *and* $x_2$ *are two linearly independent solutions to* (5.15)*. Then their Casoration* $W(n) \neq 0$ *for all* $n \in \mathbf{N}_0$*.*

*Proof.* Assume that $W(0) = 0$. Then the columns in the matrix

$$\begin{bmatrix} x_1(0) & x_2(0) \\ x_1(1) & x_2(1) \end{bmatrix}$$

are linearly dependent, and we can find $\alpha \in \mathbf{R}$ such that $x_1(0) = \alpha x_2(0)$ and $x_1(1) = \alpha x_2(1)$ (or $x_2(0) = \alpha x_1(0)$ and $x_2(1) = \alpha x_1(1)$). Let $x = x_1 - \alpha x_2$. Then $x$ is a solution to (5.15) and satisfies $x(0) = 0$, $x(1) = 0$. Thus by Theorem 5.9 we have $x_1 - \alpha x_2 = 0$, contradicting the linear independence of $x_1$ and $x_2$. Thus we must have $W(0) \neq 0$. It follows from Lemma 5.5 and the assumption $c \neq 0$ that $W(n) \neq 0$ for all $n \in \mathbf{N}_0$. □

**Theorem 5.11.** *Let* $y$ *be a real solution to*

$$x(n+2) + bx(n+1) + cx(n) = 0, \quad b, c \in \mathbf{R},\ c \neq 0, \quad n \in \mathbf{N}_0. \tag{5.20}$$

*Let* $x_1$ *and* $x_2$ *be two real linearly independent solutions to this equation. Then there exist* $c_1, c_2 \in \mathbf{R}$*, such that*

$$y(n) = c_1x_1(n) + c_2x_2(n), \quad n \in \mathbf{N}_0. \tag{5.21}$$

*Proof.* Consider the system of linear equations

$$\begin{bmatrix} x_1(0) & x_2(0) \\ x_1(1) & x_2(0) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}. \tag{5.22}$$

By Lemma 5.10 the Casoration of $x_1$ and $x_2$ satisfies $W(0) \neq 0$. Thus the equation (5.22) has a unique solution, which we denote by $\left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$. Let $u = c_1x_1 + c_2x_2 - y$. Then we have that $u$ is a solution to (5.15) and satisfies $u(0) = 0$, $u(1) = 0$. The uniqueness result implies that $u = 0$. Thus we have shown that $y = c_1x_1 + c_2x_2$. □

## 5.2 The constant coefficient case: Inhomogeneous equation

We now try to solve the inhomogeneous equation

$$x(n+2)+bx(n+1)+cx(n)=f(n),\quad b,c\in\mathbf{R},\ c\neq 0,\quad n\in\mathbf{N}_0. \tag{5.23}$$

Here $f$ is a given sequence, where we assume $f\neq 0$. First we show that to find all solutions to the equation (5.23) it suffices to find one solution, which we call a *particular solution* and then use our knowledge of the corresponding homogeneous equation, stated in Theorem 5.7.

**Theorem 5.12.** *Let $x_{\mathrm{p}}$ be a solution to* (5.23)*. Let $x_1$ and $x_2$ be two linearly independent solutions to the corresponding homogeneous equation. Then all solutions to* (5.23) *are given by*

$$x=c_1x_1+c_2x_2+x_{\mathrm{p}},\quad c_1,c_2\in\mathbf{R}. \tag{5.24}$$

*Proof.* Let $x=c_1x_1+c_2x_2+x_{\mathrm{p}}$. Then we have

$$\begin{aligned} x(n+2)+bx(n+1)+cx(n) &= c_1x_1(n+2)+c_2x_2(n+2)+x_{\mathrm{p}}(n+2)\\ &\quad +b\big(c_1x_1(n+1)+c_2x_2(n+1)+x_{\mathrm{p}}(n+1)\big)\\ &\quad +c\big(c_1x_1(n)+c_2x_2(n)+x_{\mathrm{p}}(n)\big)\\ &= c_1\big(x_1(n+2)+bx_1(n+1)+cx_1(n)\big)\\ &\quad +c_2\big(x_2(n+2)+bx_2(n+1)+cx_2(n)\big)\\ &\quad +x_{\mathrm{p}}(n+2)+bx_{\mathrm{p}}(n+1)+cx_{\mathrm{p}}(n)\\ &= c_10+c_20+f(n)=f(n). \end{aligned}$$

Thus all sequences of the form (5.24) are solutions to (5.23).

Now let $y$ be a solution to (5.23) and let $u=y-x_{\mathrm{p}}$. Then we have

$$\begin{aligned} u(n+2)+bu(n+1)+cu(n) &= y(n+2)-x_{\mathrm{p}}(n+2)+b\big(y(n+1)-x_{\mathrm{p}}(n+1)\big)\\ &\quad +c\big(y(n)-x_{\mathrm{p}}(n)\big)\\ &= y(n+2)+by(n+1)+cy(n)\\ &\quad -\big(x_{\mathrm{p}}(n+2)+bx_{\mathrm{p}}(n+1)+cx_{\mathrm{p}}(n)\big)\\ &= f(n)-f(n)=0. \end{aligned}$$

Thus $u$ is a solution to the corresponding homogeneous equation. It follows from Theorem 5.11 that there exist $c_1,c_2\in\mathbf{R}$, such that $u=c_1x_1+c_2x_2$, or $y=c_1x_1+c_2x_2+x_{\mathrm{p}}$. □

As a consequence of the above result we are left with the problem of finding a particular solution to a given inhomogeneous equation. There are no completely general methods, and, in general, the solution cannot be found in closed form. There are some techniques available, and we will present some of them. One of them is based on a simple idea. One tries to guess a solution. More precisely, if the right had side is in the form of a linear combination of functions of the form

$$r^n,\quad r^n\cos(an),\quad \text{or}\quad r^n\sin(an),$$

then the method may succeed. Here $r$ and $a$ are constants, inferred from the given right hand side. We will start with some examples to clarify the method.

**Example 5.13.** We will find the complete solution to the equation

$$x(n+2) + 2x(n+1) - 3x(n) = 4 \cdot 2^n.$$

We first solve the corresponding homogeneous equation

$$x(n+2) + 2x(n+1) - 3x(n) = 0.$$

The characteristic equation is $r_2 + 2r - 3 = 0$ with solutions $r_1 = 1$ and $r_2 = -3$. Thus the complete solution is

$$y(n) = c_1 + c_2(-3)^n, \quad c_1, c_2 \in \mathbf{R}.$$

To find one solution to the inhomogeneous equation we use the guess $u(n) = c2^n$. We insert into the equation to determine $c$. We get

$$c2^{n+2} + 2c2^{n+1} - 3c2^n = 4 \cdot 2^n.$$

This leads to $c2_2 + 2c2_1 - 3c = 4$ or $c = \frac{4}{5}$. Thus a particular solution is $y_{\mathrm{p}}(n) = \frac{4}{5}2^n$. The complete solution is then

$$x(n) = c_1 + c_2(-3)^n + \frac{4}{5}2^n, \quad c_1, c_2 \in \mathbf{R}.$$

**Example 5.14.** We will find the complete solution to the equation

$$x(n+2) + 4x(n) = \cos(2n). \tag{5.25}$$

We first solve the corresponding homogeneous equation

$$x(n+2) + 4x(n) = 0.$$

The characteristic equation is $r_2 + 4 = 0$, with solutions $r_\pm = \pm i2$. We use Theorem 5.7(iii). We have $\rho = |r_+| = 2$ and $\theta = \pi/2$. Thus the solution to the homogeneous equation is

$$y(n) = c_1 2^n \cos(\tfrac{\pi}{2}n) + c_2 2^n \sin(\tfrac{\pi}{2}n).$$

If we try to find a particular solution of the form $u(n) = c\cos(2n)$, we find after substitution into the equation a term containing $\sin(2n)$. Thus the right form is $u(n) = c\cos(2n) + d\sin(2n)$. We insert this expression into the left hand side of (5.25), and then use the addition formulas to get the following result.

$$\begin{aligned}
u(n+2) + 4u(n) &= c\cos(2(n+2)) + d\sin(2(n+2)) + 4(c\cos(2n) + d\sin(2n)) \\
&= c(\cos(2n)\cos(4) - \sin(2n)\sin(4)) \\
&\quad + d(\sin(2n)\cos(4) + \cos(2n)\sin(4)) \\
&\quad + 4(c\cos(2n) + d\sin(2n)) \\
&= (c\cos(4) + d\sin(4) + 4c)\cos(2n) \\
&\quad + (-c\sin(4) + d\cos(4) + 4d)\sin(2n)
\end{aligned}$$

Thus to solve (5.25) we have to determine $c$ and $d$, such that

$$(c\cos(4) + d\sin(4) + 4c)\cos(2n) + (-c\sin(4) + d\cos(4) + 4d)\sin(2n) = \cos(2n)$$

for all $n \in \mathbf{N}_0$. We now use that the sequences $\cos(2n)$ and $\sin(2n)$ are linearly independent. Thus we get the linear system of equations

$$\begin{aligned} c(4+\cos(4)) + d\sin(4) &= 1, \\ c(-\sin(4)) + d(4+\cos(4)) &= 0. \end{aligned}$$

The solution is

$$c = \frac{4+\cos(4)}{17+8\cos(4)}, \quad d = \frac{\sin(4)}{17+8\cos(4)}.$$

Thus the complete solution to (5.25) is given by

$$x(n) = c_1 2^n \cos(\tfrac{\pi}{2}n) + c_2 2^n \sin(\tfrac{\pi}{2}n) + \frac{4+\cos(4)}{17+8\cos(4)}\cos(2n) + \frac{\sin(4)}{17+8\cos(4)}\sin(2n).$$

**Example 5.15.** There is a different way to find a particular solution to (5.25), based on computations with complex numbers. We note that $\cos(2n) = \operatorname{Re} e^{i2n}$. We find a particular solution to the equation

$$y(n+2) + 4y(n) = e^{i2n}.$$

The particular solution to (5.25) is then found as the real part of this solution.

We note that $e^{i2n} = (e^{2i})^n$. Thus using the same technique as in Example 5.13 we guess that the solution is of the form $y(n) = ce^{2in}$, where now $c$ can be a complex constant. Insertion gives

$$\begin{aligned} y(n+2) + 4y(n) &= ce^{i(2n+4)} + 4ce^{i2n} \\ &= c(e^{4i}+4)e^{2in} = e^{2in}. \end{aligned}$$

Thus we must have

$$c = \frac{1}{e^{4i}+4} = \frac{e^{-4i}+4}{(e^{4i}+4)(e^{-4i}+4)} = \frac{e^{-4i}+4}{17+8\cos(4)}.$$

Thus the particular solution to (5.25) is given by

$$y_{\mathrm{p}}(n) = \operatorname{Re}\frac{(e^{-4i}+4)e^{2in}}{17+4\cos(4)} = \frac{4+\cos(4)}{17+8\cos(4)}\cos(2n) + \frac{\sin(4)}{17+8\cos(4)}\sin(2n).$$

This result is the same as the one in the previous example.

The method used in the three examples above is called the *method of undetermined coefficients.* As is evident from the second example, even simple right hand sides can lead to rather complicated particular solutions. To give a general prescription for the use of the method is rather complicated. We give a description of one case here.

**Method of undetermined coefficients** The method is applied to an inhomogeneous equation (5.23). There are four steps in the method:

1. Find the complete solution to the corresponding homogeneous equation in the form $x = c_1x_1 + c_2x_2$, where $x_1$ and $x_2$ are linearly independent solutions.
2. Verify that the function $x_1, x_2, f$ are linearly independent (this can be done by computing their Casoratian, or sometimes seen by inspection). If they are linearly dependent, this version of the method does not apply.

| $f(n)$ | form of $y_{\mathrm{p}}$ |
|---|---|
| $r^n$ | $cr^n$ |
| $r^n\cos(an)$ | $cr^n\cos(an)+dr^n\sin(an)$ |
| $r^n\sin(an)$ | $cr^n\cos(an)+dr^n\sin(an)$ |

Table 1: Method of undetermined coefficients

3. Verify that the right hand side is a linear combination of the functions in the left hand column of Table 1. If this is not the case, the method cannot be applied.
4. Use the form of the solution given in the second column of Table 1, insert in the inhomogeneous equation (5.23), and determine the coefficients, as in the examples.

In the case, where $x_1, x_2, f$ are linearly *dependent,* and $f$ is a linear combination of the form of functions in Table 1, the particular solution from this table is multiplied by $n$. As an example, if we instead of (5.25) consider

$$x(n+2)+4x(n)=2^n\sin(\tfrac{\pi}{2}n),$$

then the particular solution is of the form

$$cn2^n\cos(\tfrac{\pi}{2}n)+dn2^n\sin(\tfrac{\pi}{2}n),$$

or, alternatively, of the form

$$\operatorname{Im}(cn(2i)^n),$$

where in the second case $c$ may be a complex constant. One finds in both cases the particular solution

$$x_{\mathrm{p}}(n)=-\frac{n}{4}\sin(\tfrac{\pi}{2}n).$$

## 5.3 The variable coefficient case: Homogeneous equation

We now briefly look at the general homogeneous second order difference equation (5.2). As already stated, there is no general method for solving this equation. However, we can prove a general existence and uniqueness theorem.

**Theorem 5.16.** *Let $b(n)$ and $c(n)$, $n\in\mathbf{N}_0$ be real sequences. Let*

$$x(n+2)+b(n)x(n+1)+c(n)x(n)=0,\quad n\in\mathbf{N}_0. \tag{5.26}$$

*Then there exist two linearly independent solutions $x_1$ and $x_2$ to* (5.26)*. Let $x$ be any solution to* (5.26)*. Then there exist $c_1, c_2\in\mathbf{R}$, such that $x=c_1x_1+c_2x_2$. Furthermore, a solution to* (5.26) *is uniquely determined by its initial values $x(0)=y0$ and $x(1)=y1$.*

*Proof.* We define a sequence $x_1$ as follows. Let $x_1(0)=1$ and $x_1(1)=0$. Then use (5.26) to determine $x_1(2)=-b(0)x_1(1)-c(0)x_1(0)=-c(0)$, and then $x_1(3)=-b(1)x_1(2)-c(1)x_1(1)=b(1)c(0)$. In general, we determine $x_1(n)$, $n\geq 2$, from $x_1(n-1)$ and $x_1(n-2)$. Thus we get a solution $x_1$ to (5.26). A second solution $x_2$ is determined by letting $x_2(0)=0$ and $x_2(1)=1$, and then repeating the arguments above. Now we use Proposition 5.4 to show that the solutions $x_1$ and $x_2$ are linearly independent. We have

$$W(0)=\det\begin{bmatrix}x_1(0) & x_2(0)\\ x_1(1) & x_2(1)\end{bmatrix}=\det\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=1,$$

which proves the claim.

Now we prove the last statement in the theorem. Let $u$ and $v$ be solutions to (5.26), satisfying $u(0) = v(0) = y0\, u(1) = v(1) = y1$. Let $z = u - v$. Then $z(0) = 0$ and $z(1) = 0$, and (5.26) implies that $z(n) = 0$ for all $n \in \mathbf{N}_0$, such that $u = v$, as claimed. Finally, if $x$ is any solution to (5.26), then $x = x(0)x_1 + x(1)x_2$, by this uniqueness result. □

Sometimes one can guess one solution to (5.26). Then one can use the *reduction of order method* to find a second, linearly independent, solution. We state the result in the following theorem.

**Theorem 5.17** (Reduction of order)**.** *Let* $x_1$ *be a solution to* (5.26) *satisfying* $x_1(n) \neq 0$ *for all* $n \in \mathbf{N}_0$*. Then a second solution* $x_2$ *can be found by the following method. Let* $v$ *be the solution to the first order homogeneous difference equation*

$$v(n+1) + \Big(1 + b(n)\frac{x_1(n+1)}{x_1(n+2)}\Big)v(n) = 0, \quad v(0) = 1. \tag{5.27}$$

*and let* $u$ *be a solution to the first order inhomogeneous difference equation*

$$u(n+1) - u(n) = v(n). \tag{5.28}$$

*Let* $x_2(n) = u(n)x_1(n)$*. Then* $x_2$ *is a solution to* (5.26)*, and* $x_1, x_2$ *are linearly independent.*

*Proof.* Let $u$ be a sequence, and let $v = \Delta u$. Let $y(n) = u(n)x_1(n)$. Repeating the computations in (5.14), one finds immediately that in order for $y$ to solve (5.26), $y$ must be a solution to the equation in (5.27). We take the solution $v$, which satisfies the initial condition in (5.27). The existence and uniqueness of this solution follows from Theorem 3.2. Then we solve (5.28), using again Theorem 3.2, and define $x_2(n) = u(n)x_1(n)$. It remains to verify that the two solutions are linearly independent. We compute their Casoratian at zero.

$$\begin{aligned} W(0) &= \det\begin{bmatrix} x_1(0) & x_2(0) \\ x_1(1) & x_2(1) \end{bmatrix} = \det\begin{bmatrix} x_1(0) & u(0)x_1(0) \\ x_1(1) & u(1)x_1(1) \end{bmatrix} \\ &= x_1(0)x_1(1)(u(1) - u(0)) = x_1(0)x_1(1)v(0). \end{aligned}$$

By assumption $x_1(0) \neq 0$ and $x_1(1) \neq 0$, and furthermore $v(0) = 1$. Thus $x_1$ and $x_2$ are linearly independent. □

# References

[1] Stephen H. Friedberg, Arnold J. Insel, and Lawrence E. Spence, Elementary Linear Algebra: International Edition, 2/E, Pearson Higher Education, ISBN-10: 0131580345.