Lecture Notes on Difference Equations

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1 Introduction

These lecture notes are intended for the courses “Introduction to Mathematical Methods” and “Introduction to Mathematical Methods in Economics”. They contain a number of results of a general nature, and in particular an introduction to selected parts of the theory of difference equations.

2 Notation and basic concepts

The positive integers $1, 2, 3, \dots$ are denoted by $\mathbf{N}$. The non-negative integers are denoted by $\mathbf{N}_0$. All integers are denoted by $\mathbf{Z}$. The rational numbers are denoted by $\mathbf{Q}$. The real numbers are denoted by $\mathbf{R}$. We have the following obvious inclusions

$$\mathbf{N} \subset \mathbf{N}_0 \subset \mathbf{Z} \subset \mathbf{Q} \subset \mathbf{R}.$$

All inclusions are strict.

The main object of study in the theory of difference equations is sequences. A sequence of real numbers, indexed by either $\mathbf{Z}$ or $\mathbf{N}_0$, is written in either of two ways. It can be written as x_n or as $x(n)$. The second notation makes it clear that a sequence is a function from either $\mathbf{Z}$ or $\mathbf{N}_0$ to $\mathbf{R}$. *We always use the notation $x(n)$ for a sequence.*

There is one property of the set $\mathbf{N}_0$ which is important. The set is *well-ordered*, which means that any non-empty subset of $\mathbf{N}_0$ contains a smallest element.

Sums play an important role in our presentation of the results on difference equations. Here are some concrete examples.

$$1 + 2 + 3 + 4 = \sum_{n=1}^{4} n = 10 \quad \text{and} \quad 2^2 + 3^2 + 4^2 + 5^2 = \sum_{n=2}^{5} n^2 = 54.$$

In general, the structure is

$$\sum_{n=n_{\text{first}}}^{n_{\text{last}}} x(n)$$

Here n_{first} is called the lower limit and n_{last} the upper limit. $x(n)$ is called the summand. It is a function of n, which we denote by $x(n)$. Sometimes we also write it as $x(n)$ the emphasize that it is a function.

Our results are sometimes expressed as indefinite sums. Here are two examples.

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \quad \text{and} \quad \sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}.$$

One important question is how to prove such general formulas. The technique used is called *proof by induction*. We will give a description of this technique. We have a certain statement, depending on an integer $n \in \mathbf{N}$. We would like to establish its validity for all $n \in \mathbf{N}$. The proof technique comprises two steps.

1. Basic step. Prove that the statement holds for $n = 1$.

2. Induction step. Prove that if the statement holds for n, then it also holds when n is replaced by $n + 1$.

Verification of these two steps constitutes the proof of the statement for all integers $n \in \mathbf{N}$.

Let us illustrate the technique. We want to prove the formula

$$\sum_{n=1}^{N} n = \frac{N(N+1)}{2} \quad \text{for all } N \in \mathbf{N}.$$

For the first step we take $N = 1$. The formula then reads

$$1 = \frac{1(1+1)}{2},$$

which obviously is true. For the second step we assume that the formula is valid for some N and consider the left hand side for $N+1$.

$$\sum_{n=1}^{N+1} n = \Big(\sum_{n=1}^{N} n\Big) + (N+1) = \Big(\frac{N(N+1)}{2}\Big) + (N+1).$$

The second equality follows from our assumption. We now rewrite this last expression.

$$\frac{N(N+1)}{2} + N + 1 = \frac{N(N+1) + 2(N+1)}{2} = \frac{(N+1)(N+2)}{2}.$$

Thus we have shown that

$$\sum_{n=1}^{N+1} n = \frac{(N+1)((N+1)+1)}{2},$$

i.e. the formula holds with N replaced by $N+1$, and the proof is finished.

We also need a convenient notation for products. Here are two examples.

$$1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = \prod_{n=1}^{5} n = 120 \quad \text{and} \quad 3 \cdot 5 \cdot 7 \cdot 9 = \prod_{n=1}^{4} (2n+1) = 945.$$

The terminology is analogous the the one used for sums. In particular, we will be using indefinite products. The product

$$\prod_{n=1}^{N} n$$

appears so often that is has a name. It is called the factorial of N, written as $N!$. So by definition

$$N! = \prod_{n=1}^{N} n.$$

It is a number that grows rapidly with N, as can be seen in these examples.

$$\begin{aligned} 10! &= 3628800, \\ 20! &= 2432902008176640000, \\ 30! &= 265252859812191058636308480000000. \end{aligned}$$

We have the convention that

$$0! = 1.$$

The general structure of a product is

$$\prod_{n=n_{\text{first}}}^{n_{\text{last}}} x(n).$$

Important convention We use the following conventions. If $n_1 > n_2$, then by definition

$$\sum_{n=n_1}^{n_2} a(n) = 0 \quad \text{and} \quad \prod_{n=n_1}^{n_2} a(n) = 1. \tag{2.1}$$

By this convention we have that

$$\sum_{n=0}^{-1} a(n) = 0 \quad \text{and} \quad \prod_{n=0}^{-1} a(n) = 1. \tag{2.2}$$

We now introduce the binomial formula. Given $x, y \in \mathbf{R}$, we have

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}. \tag{2.3}$$

Here the *binomial coefficients* are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}, \quad k = 0, \ldots, n. \tag{2.4}$$

Recall our convention $0! = 1$. The binomial coefficients satisfy many identities. One of them is the following.

$$\binom{n+1}{k} = \binom{n}{k-1} + \binom{n}{k}, \quad k = 1, \ldots, n. \tag{2.5}$$

This result is the consequence of the following computation.

$$\begin{aligned}
\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!(n-k+1)!} + \frac{n!}{k!(n-k)!} \\
&= \frac{n!k}{k(k-1)!(n-k+1)!} + \frac{n!(n+1-k)}{k!(n-k)!(n+1-k)} \\
&= \frac{n!k + n!(n+1-k)}{k!(n+1-k)!} = \frac{(n+1)!}{k!(n+1-k)!} \\
&= \binom{n+1}{k}.
\end{aligned}$$

Exercises

Exercise 2.1. Prove by induction that we have

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}.$$

Exercise 2.2. Let $q \in \mathbf{R}$ satisfy $q \neq 1$. Prove by induction that

$$\sum_{n=0}^{N} q^n = \frac{q^{N+1}-1}{q-1}. \tag{2.6}$$

What is $\sum_{n=0}^{N} q^n$ for $q = 1$?

Exercise 2.3. Prove by induction that we have

$$\sum_{n=1}^{N} n^3 = \frac{N^2(N+1)^2}{4}.$$

Exercise 2.4. Prove that

$$\sum_{n=1}^{N} n^3 = \Big(\sum_{n=1}^{N} n\Big)^2.$$

Exercise 2.5. Prove (2.3).

Exercise 2.6. Prove the following result

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Exercise 2.7. Prove the following result

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0.$$

3 First order difference equations

In many cases it is of interest to model the evolution of some system over time. There are two distinct cases. One can think of time as a continuous variable, or one can think of time as a discrete variable. The first case often leads to differential equations. We will not discuss differential equations in these notes.

We consider a time period T and observe (or measure) the system at times $t = nT$, $n \in \mathbf{N}_0$. The result is a sequence $x(0), x(1), x(2), \ldots$. In some cases these values are obtained from a function f, which is defined for all $t \geq 0$. In this case $x(n) = f(nT)$. This method of obtaining the values is called periodic sampling. One models the system using a difference equation, or what is sometimes called a recurrence relation.

In this section we will consider the simplest cases first. We start with the following equation

$$x(n+1) = ax(n), \quad n \in \mathbf{N}_0, \tag{3.1}$$

where a is a given constant. The solution is given by

$$x(n) = a^n x(0). \tag{3.2}$$

The value $x(0)$ is called the *initial value*. To prove that (3.2) solves (3.1), we compute as follows.

$$x(n+1) = a^{n+1}x(0) = a(a^n x(0)) = ax(n).$$

Example 3.1. An amount of USD10,000 is deposited in a bank account with an annual interest rate of 4%. Determine the balance of the account after 15 years. This problem leads to the difference equation

$$b(n+1) = 1.04b(n), \quad b(0) = 10,000.$$

The solution is

$$b(n) = (1.04)^n 10,000,$$

in particular $b(15) = 18,009.44$.

We write the equation (3.1) as

$$x(n+1) - ax(n) = 0. \tag{3.3}$$

This equation is called a homogeneous first order difference equation with constant coefficients. The term homogeneous means that the right hand side is zero. A corresponding inhomogeneous equation is given as

$$x(n+1) - ax(n) = c, \tag{3.4}$$

where we take the right hand side to be a constant different from zero.

The equation (3.3) is called linear, since it satisfies the *superposition principle.* Let $y(n)$ and $z(n)$ be two solutions to (3.3), and let $\alpha, \beta \in \mathbf{R}$ be two real numbers. Define $w(n) = \alpha y(n) + \beta z(n)$. Then $w(n)$ also satisfies (3.3), as the following computation shows.

$$\begin{aligned} w(n+1) - aw(n) &= \alpha y(n+1) + \beta z(n+1) - a(\alpha y(n) + \beta z(n)) \\ &= \alpha(y(n+1) - ay(n)) + \beta(z(n+1) - az(n)) = \alpha 0 + \beta 0 = 0. \end{aligned}$$

We now solve (3.4). The idea is to compute a number of terms, guess the structure of the solution, and then prove that we have indeed found the solution. First we compute a number of terms. In the computation of $x(2)$ we give all intermediate steps. These are omitted in the computation of $x(3)$ etc.

$$\begin{aligned} x(1) &= ax(0) + c, \\ x(2) &= ax(1) + c = a(ax(0) + c) + c = a^2x(0) + ac + c, \\ x(3) &= ax(2) + c = a^3x(0) + a^2c + ac + c, \\ x(4) &= ax(3) + c = a^4x(0) + a^3c + a^2c + ac + c, \\ x(5) &= ax(4) + c = a^5x(0) + a^4c + a^3c + a^2c + ac + c, \\ &\vdots \\ x(n) &= a^nx(0) + c\sum_{k=0}^{n-1} a^k. \end{aligned}$$

Thus we have guessed that the solution is given by

$$x(n) = a^nx(0) + c\sum_{k=0}^{n-1} a^k. \tag{3.5}$$

To prove that (3.5) is a solution to (3.4), we must prove (3.5) satisfies this equation. We compute as follows.

$$\begin{aligned} x(n+1) &= a^{n+1}x(0) + c\sum_{k=0}^{n} a^k \\ &= a^{n+1}x(0) + c(1 + a + a_2 + \cdots + a^{n-1} + a^n) \\ &= a(a^nx(0)) + c + a(c(1 + a + a_2 + \cdots + a^{n-1})) \\ &= a\Big(a^nx(0) + c\sum_{k=0}^{n-1} a^k\Big) + c \\ &= ax(n) + c. \end{aligned}$$

Thus we have shown that (3.5) is a solution to (3.4). For $a \neq 1$ the solution (3.5) can be rewritten using the result (2.6):

$$x(n) = a^n x(0) + c\frac{a^n - 1}{a - 1}. \tag{3.6}$$

In the general case both a and c will be functions of n. We have the following result.

Theorem 3.2. *Let $a(n)$, and $c(n)$, $n \in \mathbf{N}_0$, be real sequences. Then the linear first order difference equation*

$$x(n+1) = a(n)x(n) + c(n) \quad \textit{with initial condition } x(0) = y_0 \tag{3.7}$$

has the solution

$$y(n) = \left(\prod_{k=0}^{n-1} a(k)\right) y_0 + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n-1} a(j)\right) c(k). \tag{3.8}$$

The solution is unique.

Proof. We define the sequence $y(n)$ by (3.8). We must show that it satisfies the equation (3.7) and the initial condition. Due to the convention (2.1) the initial condition is trivially satisfied. We first write out the expression for $y(n+1)$

$$y(n+1) = \left(\prod_{k=0}^{n} a(k)\right) y_0 + \sum_{k=0}^{n}\left(\prod_{j=k+1}^{n} a(j)\right) c(k).$$

We then rewrite the last term above as follows, using (2.1).

$$\begin{aligned}\sum_{k=0}^{n}\left(\prod_{j=k+1}^{n} a(j)\right) c(k) &= \prod_{j=n+1}^{n} a(j)c(n) + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n} a(j)\right) c(k)\\ &= c(n) + \sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n} a(j)\right) c(k) = c(n) + a(n)\sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n-1} a(j)\right) c(k).\end{aligned}$$

Using this result we get

$$y(n+1) = a(n)\left(\prod_{k=0}^{n-1} a(k)\right) y(0) + c(n) + a(n)\sum_{k=0}^{n-1}\left(\prod_{j=k+1}^{n-1} a(j)\right) c(k),$$

which implies

$$y(n+1) = a(n)y(n) + c(n).$$

Thus we have shown that $y(n)$ is a solution. Finally we must prove uniqueness. Assume that we have two solutions $y(n)$ and $\tilde{y}(n)$, which satisfy (3.7), i.e. both the equation and the initial condition are satisfied by both solutions. Now consider $\{n \in \mathbf{N}_0 \mid y(n) \neq \tilde{y}(n)\}$. Let n_0 be the smallest integer in this set. Assume $n_0 \geq 1$. By the definition of n_0 we have $y(n_0 - 1) = \tilde{y}(n_0 - 1)$, and then

$$y(n_0) = a(n_0 - 1)y(n_0 - 1) + c(n_0 - 1) = a(n_0 - 1)\tilde{y}(n_0 - 1) + c(n_0 - 1) = \tilde{y}(n_0),$$

which is a contradiction. Thus we must have $n_0 = 0$. But $y(0) = \tilde{y}(0)$, since the two equations satisfy the same initial condition. It follows that the solution is unique. $\square$

3.1 Examples

We now give some examples. Details should be worked out by the reader.

Example 3.3. Consider the problem

$$x(n+1) = -x(n), \quad x(0) = 3.$$

Using (3.5) with $c = 0$ we get the solution

$$x(n) = (-1)^n 3.$$

Now consider the inhomogeneous problem

$$x(n+1) = -x(n) + 4, \quad x(0) = 3.$$

Using (3.6) we get the solution

$$x(n) = (-1)^n 3 - 2\big((-1)^n - 1\big) = (-1)^n + 2.$$

Example 3.4. Consider the problem

$$x(n+1) = 2x(n) + n, \quad x(0) = 5.$$

Using the general formula (3.8) we get the solution

$$x(n) = 5 \cdot 2^n + \sum_{k=0}^{n-1} k 2^{n-1-k} = 5 \cdot 2^n + 2^n - n - 1.$$

The last equality requires results that are not covered by this course, so the first expression is sufficient as the solution to the problem.

Example 3.5. Consider the problem

$$x(n+1) = (n-4)x(n), \quad x(0) = 1. \tag{3.9}$$

This problem can be solved in two different manners. One can directly use the general formula (3.8). In this case one gets the solution

$$x(n) = \prod_{k=0}^{n-1} (k-4).$$

But this solution is not very explicit. A more explicit solution can be found by noting that for $n \geq 5$ the product contains the factor 0, hence the product is zero. Thus one has the explicit solution:

$$x(n) = \begin{cases} 1 & n = 0, \\ -4 & n = 1, \\ 12 & n = 2, \\ -24 & n = 3, \\ 24 & n = 4, \\ 0 & n \geq 5. \end{cases} \tag{3.10}$$

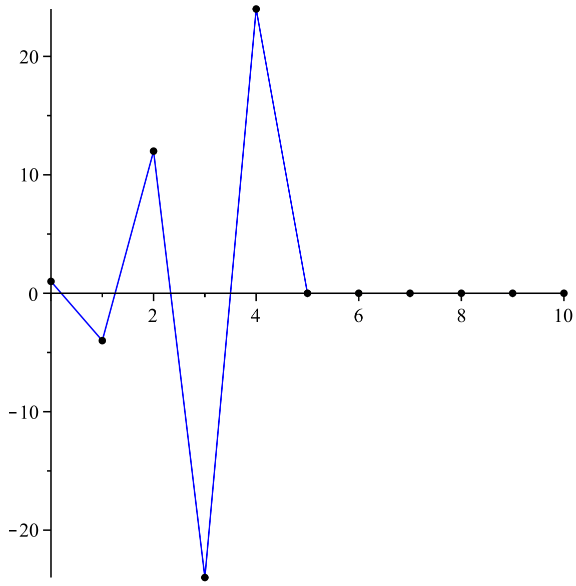

Figure 3.1: Point plot of the solution (3.10). Points connected with blue lines

We illustrate the solution in Figure 3.1. Here we plot the values of $x(n)$ as filled circles, connected by blue line segments. We include the line segments to visualize the variations in the values.

We note that the solution (3.10) is very sensitive to small changes in the equation. If we add a small constant inhomogeneous term, the solution will rapidly diverge from the solution zero for $n \geq 5$. As an example we consider

$$x(n+1) = (n-4)x(n) + \frac{1}{20}, \quad x(0) = 1. \tag{3.11}$$

A plot of this solution is shown in Figure 3.2.

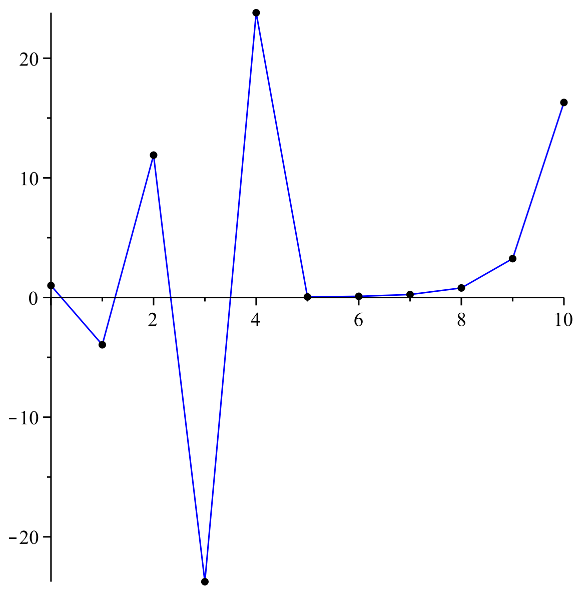

Figure 3.2: Point plot of the solution to (3.11). Points connected with blue lines

Example 3.6. (**Note:** In version 2 of the notes this was Example 3.3.) Let us consider the payment of a loan. Payments are made periodically, e.g. once a month. The interest rate per period is $100r\%$. The payment at the end of each period is denoted $p(n)$. The initial loan is $q(0)$. The outstanding balance after n payments is denoted $q(n)$. Thus $q(n)$ must satisfy the difference equation

$$q(n+1) = (1+r)q(n) - p(n). \tag{3.12}$$

The solution follows from (3.8).

$$q(n) = (1+r)^n q(0) - \sum_{k=0}^{n-1} (1+r)^{n-k-1} p(k). \tag{3.13}$$

Often the loan is paid back in equal installments, i.e. $p(n) = p$ for all n. Then the above sum can be computed. We get the result

$$q(n) = (1+r)^n q(0) - ((1+r)^n - 1)\frac{p}{r}. \tag{3.14}$$

Suppose that we want to pay back the loan in N installments. Then the installment is determined by

$$p = q(0)\frac{r}{1-(1+r)^{-N}} \tag{3.15}$$

Exercises

Exercise 3.1. Fill in the details in Example 3.6. In particular the computations leading to (3.14).

Exercise 3.2. Discuss the applications of the results in Example 3.6.

Exercise 3.3. Adapt the results in the Example 3.6 to the case, where initially no installments are paid.

Exercise 3.4. Discuss the application to loans with a variable interest rate of the results in this section.

Exercise 3.5. Implement the various formulas for interest computation and loan amortization on a programmable calculator or in Maple. In particular, implement the formulas for loans with a variable interest rate and try them out on some real world examples.

4 Difference calculus

Before we proceed to the study of general difference equations, we establish some results on the difference calculus. We denote all functions from $\mathbf{Z}$ to $\mathbf{R}$ by $\mathsf{S}(\mathbf{Z})$, and all functions from $\mathbf{N}_0$ to $\mathbf{R}$ by $\mathsf{S}(\mathbf{N}_0)$.

The set $\mathsf{S}(\mathbf{Z})$ is a real vector space. See [3] for the definition.

Proposition 4.1. *The set* $\mathsf{S}(\mathbf{Z})$ *is a real vector space, if the addition is defined as*

$$(x+y)(n) = x(n) + y(n), \quad x, y \in \mathsf{S}(\mathbf{Z}),$$

and the scalar multiplication as

$$(ax)(n) = ax(n), \quad a \in \mathbf{R},\ x \in \mathsf{S}(\mathbf{Z}).$$

Below we give definitions and results for $x \in \mathsf{S}(\mathbf{Z})$. To apply these results to functions (sequences) on $\mathbf{N}_0$, we consider $\mathsf{S}(\mathbf{N}_0)$ as a subset of $\mathsf{S}(\mathbf{Z})$. This is done in the following manner. Given $x \in \mathsf{S}(\mathbf{N}_0)$, we define

$$(\iota x)(n) = \begin{cases} x(n) & \text{for } n \geq 0, \\ 0 & \text{for } n < 0. \end{cases}$$

A function that maps a function $x(n)$ to a new function $y(n)$ is called an *operator*. An example is the operator $\iota\colon \mathsf{S}(\mathbf{N}_0) \to \mathsf{S}(\mathbf{Z})$ defined above. We define the operators Δ, S, and I as follows:

Definition 4.2. The *shift operator* $S\colon \mathsf{S}(\mathbf{Z}) \to \mathsf{S}(\mathbf{Z})$ is defined by

$$(Sx)(n) = x(n+1). \tag{4.1}$$

The *difference operator* Δ is defined by

$$(\Delta x)(n) = x(n+1) - x(n). \tag{4.2}$$

The *identity operator* I is defined by

$$(Ix)(n) = x(n). \tag{4.3}$$

The relation between the three operators is

$$\Delta = S - I. \tag{4.4}$$

The operators S and Δ are linear. We recall from [3] that an operator $U\colon \mathsf{S}(\mathbf{Z}) \to \mathsf{S}(\mathbf{Z})$ is said to be linear, if it satisfies

$$U(x+y) = Ux + Uy \quad \text{for all } x, y \in \mathsf{S}(\mathbf{Z}), \tag{4.5}$$

$$U(ax) = aUx \quad \text{for all } x \in \mathsf{S}(\mathbf{Z}) \text{ and } a \in \mathbf{R}. \tag{4.6}$$

We recall that composition of two linear operators $U, V\colon \mathsf{S}(\mathbf{Z}) \to \mathsf{S}(\mathbf{Z})$ is defined as $(U \circ Vx)(n) = (U(Vx))(n)$. If $U = V$, we write $U \circ U = U^2$. Usually we also write UV instead of $U \circ V$.

5 Second order linear difference equations

We will now present the theory of second order linear difference equations. In contrast to the first order case, there is no general formula that gives the solution to all such equations. One has to impose additional conditions in order to get a general formula.

The general form of a second order linear difference equation is

$$x(n+2) + b(n)x(n+1) + c(n)x(n) = f(n), \quad n \in \mathbf{N}_0. \tag{5.1}$$

Here $b(n)$, $c(n)$, $f(n)$ are given sequences. If $f(n) = 0$ for all n, then the equation is homogeneous, viz.

$$x(n+2) + b(n)x(n+1) + c(n)x(n) = 0, \quad n \in \mathbf{N}_0. \tag{5.2}$$

If we define the operator

$$(Lx)(n) = x(n+2) + b(n)x(n+1) + c(n)x(n),$$

then $L\colon \mathsf{S}(\mathbf{N}_0) \to \mathsf{S}(\mathbf{N}_0)$ is a linear operator, see Section 5.5.

We need some techniques and results from linear algebra in order to discuss the second and higher order equations.

Definition 5.1. Let $x_j \in \mathsf{S}(\mathbf{N}_0)$, $j = 1,\dots,N$. The list of vectors $x_1, x_2, \dots, x_N$ is said to be *linearly independent*, if for all $c_1, c_2, \dots, c_N$

$$c_1x_1 + c_2x_2 + \cdots + c_Nx_N = 0 \quad \text{implies} \quad c_1 = 0, c_2 = 0, \dots, c_N = 0. \tag{5.3}$$

If the list of vectors is not linearly independent, it is said to be *linearly dependent*.

Remark 5.2. We make a number of remarks on this definition.

(i) The definition is the same as in [3], and many of the results stated there carry over to the present more abstract framework.

(ii) We call the collection of vectors $x_1, x_2, \dots, x_N$ a list, since the elements are viewed as ordered. In particular, in contrast to a set, repetition of entries is significant.

(iii) Let us state explicitly what it means that the list of vectors $x_1, x_2, \dots, x_N$ is *linearly dependent*. It means that there exist $c_1, c_2, \dots, c_N$ with at least one $c_j \neq 0$, such that

$$c_1x_1(n) + c_2x_2(n) + \cdots + c_Nx_N(n) = 0 \quad \text{for all } n \in \mathbf{N}_0. \tag{5.4}$$

We will need some results to prove linear independence of vectors in $\mathsf{S}(\mathbf{N}_0)$. We give the general definition here. In this section we use it only for $N = 2$.

Definition 5.3. Let $N \geq 2$. Let $x_1, x_2, \dots, x_N \in \mathsf{S}(\mathbf{N}_0)$. Then we define the *Casoratian* by

$$W(n) = \begin{vmatrix} x_1(n) & x_2(n) & \cdots & x_N(n) \\ x_1(n+1) & x_2(n+1) & \cdots & x_N(n+1) \\ \vdots & \vdots & \ddots & \vdots \\ x_1(n+N-1) & x_2(n+N-1) & \cdots & x_N(n+N-1) \end{vmatrix} \tag{5.5}$$

Note that the Casoratian is a function of n. It also depends on the vectors $x_1, x_2, \dots, x_N$, but this is not made explicit in the notation.

The Casoratian gives us a convenient method to determine, whether a given set of vectors is linearly independent.

Proposition 5.4. *Let $N \geq 2$. Let $x_1, x_2, \dots, x_N \in \mathsf{S}(\mathbf{N}_0)$. If there exists an $n_0 \in \mathbf{N}_0$, such that $W(n_0) \neq 0$, then $x_1, x_2, \dots, x_N$ are linearly independent.*

Proof. We give the proof in the case $N = 2$. Thus we have sequences x_1, x_2, and $n_0 \in \mathbf{N}_0$, such that

$$W(n_0) = \begin{vmatrix} x_1(n_0) & x_2(n_0) \\ x_1(n_0+1) & x_2(n_0+1) \end{vmatrix} \neq 0. \tag{5.6}$$

Now assume that we have a linear combination

$$c_1x_1 + c_2x_2 = 0.$$

More explicitly, this means that $c_1x_1(n)+c_2x_2(n) = 0$ for all $n \in \mathbf{N}_0$. In particular, we have

$$\begin{aligned} c_1x_1(n_0) + c_2x_2(n_0) &= 0, \\ c_1x_1(n_0+1) + c_2x_2(n_0+1) &= 0. \end{aligned}$$

But then $c_1 = c_2 = 0$, by well-known results from linear algebra, see [3].

Let us explain in some detail how we use the results from [3] to get this result. We write the two linear equations in matrix form,

$$\begin{bmatrix} x_1(n_0) & x_2(n_0) \\ x_1(n_0+1) & x_2(n_0+1) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Now the determinant condition $W(n_0) \neq 0$ implies that the coefficient matrix is invertible, hence the only solution is the trivial one, $c_1 = 0$ and $c_2 = 0$.

The general case is left as an exercise. □

Lemma 5.5. *Assume that x_1 and x_2 are two solution to the homogeneous equation* (5.2)*. Let $W(n)$ be the Casoration of these solutions, given by* (5.5)*, $N = 2$. Then we have for $n_0 \in \mathbf{N}_0$ that for all $n \geq n_0$*

$$W(n) = W(n_0) \prod_{k=n_0}^{n-1} c(n). \tag{5.7}$$

Proof. The equation (5.2) implies

$$x_j(n+2) = -c(n)x_j(n) - b(n)x_j(n+1).$$

Then we have

$$\begin{aligned} W(n+1) &= \begin{vmatrix} x_1(n+1) & x_2(n+1) \\ x_1(n+2) & x_2(n+2) \end{vmatrix} \\ &= \begin{vmatrix} x_1(n+1) & x_2(n+1) \\ -c(n)x_1(n) - b(n)x_1(n+1) & -c(n)x_2(n) - b(n)x_2(n+1) \end{vmatrix} \\ &= \begin{vmatrix} x_1(n+1) & x_2(n+1) \\ -c(n)x_1(n) & -c(n)x_2(n) \end{vmatrix} = c(n)W(n). \end{aligned}$$

Solving the linear first order difference equation $W(n+1) = c(n)W(n)$ with initial value $W(n_0)$ (see Theorem 3.2), we conclude the proof. □

5.1 The constant coefficient case: Homogeneous equation

In this case the functions $b(n)$ and $c(n)$ are constants, denoted by b and c. We start by solving the homogeneous equation. Thus we consider the equation

$$x(n+2) + bx(n+1) + cx(n) = 0, \quad n \in \mathbf{N}_0, \quad \text{with } b, c \in \mathbf{R}. \tag{5.8}$$

We now go through the steps leading to the complete solution to this equation, and then at the end we summarize the results in a theorem.

We assume that $c \neq 0$, since otherwise the equation is a first order equation for the function $y(n) = x(n+1)$, which we have already solved. To solve the equation (5.8) we try to find solutions of the form $x(n) = r^n$, where $r \neq 0$, and r may be either real or complex.

We will see below why we have to allow complex solutions. Insert $x(n) = r^n$ into (5.8) and use $r \neq 0$ to get the equation

$$r^2 + br + c = 0. \tag{5.9}$$

This equation is called the *characteristic equation* of (5.8).

There are now three possibilities.

Case 1 If $b^2 - 4c > 0$, then (5.9) has two different real roots, which we denote by r_1 and r_2.

Case 2 If $b^2 - 4c = 0$, then (5.9) has a real double root, which we denote by r_0.

Case 3 If $b^2 - 4c < 0$, then (5.9) has pair of complex conjugate roots, which we denote by $r_\pm = \alpha \pm i\beta$, $\beta > 0$.

Consider first **Case 1**. Let $x_1(n) = r_1^n$ and $x_2(n) = r_2^n$, $n \in \mathbf{N}_0$. We now use Proposition 5.4 with $n_0 = 0$. We have

$$W(0) = \begin{vmatrix} r_1^0 & r_2^0 \\ r_1^1 & r_2^1 \end{vmatrix} = r_2 - r_1 \neq 0.$$

Thus we have found two linearly independent solutions to (5.8). Note that the solutions are real.

Next we consider **Case 3**. Since we assume that the coefficients in (5.8) are real, we would like to find real solutions. We state the following result.

Proposition 5.6. *Let y be a complex solution to* (5.8)*. Then $x_1(n) = \operatorname{Re} y(n)$ and $x_2(n) = \operatorname{Im} y(n)$ are real solutions to* (5.8)*.*

Proof. By assumption we have that

$$y(n+2) + by(n+1) + cy(n) = 0 \quad \text{for all } n \in \mathbf{N}_0.$$

Taking the real part and using that b, c are real, we get

$$\operatorname{Re} y(n+2) + b \operatorname{Re} y(n+1) + c \operatorname{Re} y(n) = 0 \quad \text{for all } n \in \mathbf{N}_0,$$

which proves the result for x_1. The proof for x_2 follows in the same manner by taking imaginary parts. □

We now use some results concerning complex numbers, see [3, Appendix C] and also [1]. We know that $y(n) = r_+^n$ is a solution, and we use Proposition 5.6 to find two real solutions, given by $x_1(n) = \operatorname{Re} r_+^n$ and $x_1(n) = \operatorname{Im} r_+^n$. We now rewrite these two solutions. Let

$$\rho = |r_+| = \sqrt{\alpha^2 + \beta^2} \quad \text{and} \quad \theta = \operatorname{Arg} r_+. \tag{5.10}$$

We recall that we have $0 < \theta < \pi$, since we have $\beta > 0$. Now $r_+ = \rho e^{i\theta}$ and then $r_+^n = \rho^n e^{in\theta}$. Taking real and imaginary parts and using the de Moivre formula, we get

$$x_1(n) = \rho^n \cos(n\theta) \quad \text{and} \quad x_2(n) = \rho^n \sin(n\theta). \tag{5.11}$$

We use Proposition 5.4 to verify that x_1 and x_2 are linearly independent. We have

$$W(0) = \begin{vmatrix} 1 & 0 \\ \rho\cos(\theta) & \rho\sin(\theta) \end{vmatrix} = \rho\sin(\theta) \neq 0,$$

since $\rho > 0$ and $0 < \theta < \pi$.

It remains to consider **Case 2**. We have one real solution x_1 given by $x_1(n) = r_0^n$. We note that $r_0 = -\frac{b}{2}$. We need to find another solution. To do this we use a general procedure known as *reduction of order*. We try to find the second solution in the form $y(n) = u(n)x_1(n)$. Using the notation $(\Delta u)(n) = u(n+1) - u(n)$, see (4.2), we have

$$y(n+1) = u(n)x_1(n+1) + (\Delta u)(n)x(n+1), \tag{5.12}$$

$$y(n+2) = u(n)x_1(n+2) + (\Delta u)(n)x_1(n+2) + (\Delta u)(n+1)x_1(n+2). \tag{5.13}$$

We now compute as follows, using $x_1(n+2) + bx_1(n+1) + cx_1(n) = 0$,

$$\begin{aligned} y(n+2) &+ by(n+1) + cy(n) \\ &= u(n)x_1(n+2) + (\Delta u)(n)x_1(n+2) + (\Delta u)(n+1)x_1(n+2) \\ &\quad + b\big(u(n)x_1(n+1) + (\Delta u)(n)x_1(n+1)\big) \\ &\quad + c\big(u(n)x_1(n)\big) \\ &= (\Delta u)(n+1)x_1(n+2) + (\Delta u)(n)\big(x_1(n+2) + bx_1(n+1)\big). \end{aligned} \tag{5.14}$$

Now we look for $y(n)$ satisfying $y(n+2) + by(n+1) + cy(n) = 0$. Using $x_1(n) = r_0^n$, we get from (5.14) after division by $x_1(n+2)$ the equation

$$(\Delta u)(n+1) + (\Delta u)(n)\big(1 + b\frac{x_1(n+1)}{x_1(n+2)}\big) = (\Delta u)(n+1) + (\Delta u)(n)\big(1 + b\frac{1}{r_0}\big) = 0.$$

We have

$$1 + b\frac{1}{r_0} = 1 + b\frac{1}{-\frac{b}{2}} = -1.$$

Solving the first order difference equation $(\Delta u)(n+1) - (\Delta u)(n) = 0$, we get $(\Delta u)(n) = c_1$, and then solving the first order equation $u(n+1) - u(n) = c_1$, we get

$$u(n) = c_1 n + c_2, \quad c_1, c_2 \in \mathbf{R}.$$

Thus we have found the solutions $y(n) = (c_1 n + c_2)r_0^n$. $c_1 = 0$ leads to the already known solutions $c_2 r_0^n$, so we take $c_2 = 0$ and $c_1 = 1$ to get the solution $x_2(n) = nr_0^n$. We compute the Casoration at zero of the two solutions that we have found.

$$W(0) = \begin{vmatrix} 1 & 0 \\ r_0 & 1r_0 \end{vmatrix} = r_0 \neq 0.$$

Thus we have found two linearly independent solutions.

We summarize the above results in the following Theorem.

Theorem 5.7. *The second order homogeneous difference equation with constant real coefficients*

$$x(n+2) + bx(n+1) + cx(n) = 0, \quad b, c \in \mathbf{R},\ c \neq 0, \quad n \in \mathbf{N}_0, \tag{5.15}$$

always has two real linearly independent solutions x_1 and x_2. They are determined from the characteristic equation

$$r^2 + br + c = 0. \tag{5.16}$$

(i) *If $b^2 - 4c > 0$, the two real solutions to* (5.16) *are denoted by r_1 and r_2. The two linearly independent solutions to* (5.15) *are given by*

$$x_1(n) = r_1^n \quad \textit{and} \quad x_2(n) = r_2^n, \quad n \in \mathbf{N}_0. \tag{5.17}$$

(ii) *If* $b^2 - 4c = 0$*, the real solution to* (5.16) *is denoted by* r_0*. The two linearly independent solutions to* (5.15) *are given by*

$$x_1(n) = r_0^n \quad \textit{and} \quad x_2(n) = nr_0^n, \quad n \in \mathbf{N}_0. \tag{5.18}$$

(iii) *If* $b^2 - 4c < 0$*, the two complex conjugate solution to* (5.16) *are denoted by* $r_\pm = \alpha \pm i\beta$*,* $\beta > 0$*. Let* $r_+ = \rho e^{i\theta} = \rho(\cos(\theta) + i\sin(\theta))$*,* $\rho = |r_+|$*,* $\theta = \operatorname{Arg} r_+$*. The two linearly independent solutions to* (5.15) *are given by*

$$x_1(n) = \rho^n \cos(n\theta) \quad \textit{and} \quad x_2(n) = \rho^n \sin(n\theta), \quad n \in \mathbf{N}_0. \tag{5.19}$$

Next we show how to describe all solutions to the equation (5.15).

Theorem 5.8 (Superposition principle)**.** *Let* x_1 *and* x_2 *be solutions to* (5.15)*. Let* $c_1, c_2 \in \mathbf{R}$*. Then* $y = c_1x_1 + c_2x_2$ *is a solution to* (5.15)*.*

Proof. The proof is left as an exercise. □

Theorem 5.9 (Uniqueness)**.** *A solution* y *to* (5.15) *is uniquely determined by the initial values* $y0 = y(0)$ *and* $y1 = y(1)$*.*

Proof. Assume that we have two solutions y_1 and y_2 to (5.15), with the initial values $y0$ and $y1$, i.e. $y_1(0) = y_2(0) = y0$ and $y_1(1) = y_2(1) = y1$. We must show that $y_1(n) = y_2(n)$ for all $n \in \mathbf{N}_0$. Let $y(n) = y_1(n) - y_2(n)$. Then by Theorem 5.8 y satisfies (5.15) with initial values zero. It follows from (5.15), written as

$$x(n+2) = -bx(n+1) - cx(n),$$

that $y(n) = 0$ for all $n \in \mathbf{N}_0$. More precisely, one proves this by induction. □

Before proving the next Theorem we need the following result, which complements Proposition 5.4.

Lemma 5.10. *Assume that* x_1 *and* x_2 *are two linearly independent solutions to* (5.15)*. Then their Casoration* $W(n) \neq 0$ *for all* $n \in \mathbf{N}_0$*.*

Proof. Assume that $W(0) = 0$. Then the columns in the matrix

$$\begin{bmatrix} x_1(0) & x_2(0) \\ x_1(1) & x_2(1) \end{bmatrix}$$

are linearly dependent, and we can find $\alpha \in \mathbf{R}$ such that $x_1(0) = \alpha x_2(0)$ and $x_1(1) = \alpha x_2(1)$ (or $x_2(0) = \alpha x_1(0)$ and $x_2(1) = \alpha x_1(1)$). Let $x = x_1 - \alpha x_2$. Then x is a solution to (5.15) and satisfies $x(0) = 0$, $x(1) = 0$. Thus by Theorem 5.9 we have $x_1 - \alpha x_2 = 0$, contradicting the linear independence of x_1 and x_2. Thus we must have $W(0) \neq 0$. It follows from Lemma 5.5 and the assumption $c \neq 0$ that $W(n) \neq 0$ for all $n \in \mathbf{N}_0$. □

Theorem 5.11. *Let* y *be a real solution to*

$$x(n+2) + bx(n+1) + cx(n) = 0, \quad b, c \in \mathbf{R},\ c \neq 0, \quad n \in \mathbf{N}_0. \tag{5.20}$$

Let x_1 *and* x_2 *be two real linearly independent solutions to this equation. Then there exist* $c_1, c_2 \in \mathbf{R}$*, such that*

$$y(n) = c_1x_1(n) + c_2x_2(n), \quad n \in \mathbf{N}_0. \tag{5.21}$$

Proof. Consider the system of linear equations

$$\begin{bmatrix} x_1(0) & x_2(0) \\ x_1(1) & x_2(1) \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} y(0) \\ y(1) \end{bmatrix}. \tag{5.22}$$

By Lemma 5.10 the Casoration of x_1 and x_2 satisfies $W(0) \neq 0$. Thus the equation (5.22) has a unique solution, which we denote by $\left[\begin{smallmatrix} c_1 \\ c_2 \end{smallmatrix}\right]$. Let $u = c_1x_1 + c_2x_2 - y$. Then we have that u is a solution to (5.15) and satisfies $u(0) = 0$, $u(1) = 0$. The uniqueness result implies that $u = 0$. Thus we have shown that $y = c_1x_1 + c_2x_2$. □

Example 5.12. Consider the homogeneous equation

$$x(n+2) - x(n+1) - x(n) = 0. \tag{5.23}$$

The characteristic equation is $r^2 - r - 1 = 0$, which has the solutions

$$r_1 = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad r_2 = \frac{1-\sqrt{5}}{2}.$$

Thus the complete solution is given by

$$x(n) = c_1\Big(\frac{1+\sqrt{5}}{2}\Big)^n + c_2\Big(\frac{1-\sqrt{5}}{2}\Big)^n, \quad c_1 \in \mathbf{R},\ c_2 \in \mathbf{R}.$$

With the initial conditions $x(0) = 0$ and $x(1) = 1$ the solution is called the *Fibonacci numbers* F_n, where

$$F_n = \frac{1}{\sqrt{5}}\bigg(\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\bigg).$$

With the initial conditions $x(0) = 2$ and $x(1) = 1$ the solution is called the *Lucas numbers* L_n, where

$$L_n = \Big(\frac{1+\sqrt{5}}{2}\Big)^n + \Big(\frac{1-\sqrt{5}}{2}\Big)^n.$$

5.2 The constant coefficient case: Inhomogeneous equation

We now try to solve the inhomogeneous equation

$$x(n+2) + bx(n+1) + cx(n) = f(n), \quad b, c \in \mathbf{R},\ c \neq 0, \quad n \in \mathbf{N}_0. \tag{5.24}$$

Here f is a given sequence, where we assume $f \neq 0$. First we show that to find all solutions to the equation (5.24) it suffices to find one solution, which we call a *particular solution* and then use our knowledge of the corresponding homogeneous equation, stated in Theorem 5.7.

Theorem 5.13. *Let x_{p} be a solution to* (5.24)*. Let x_1 and x_2 be two linearly independent solutions to the corresponding homogeneous equation. Then all solutions to* (5.24) *are given by*

$$x = c_1x_1 + c_2x_2 + x_{\mathrm{p}}, \quad c_1, c_2 \in \mathbf{R}. \tag{5.25}$$

Proof. Let $x = c_1x_1 + c_2x_2 + x_{\mathrm{p}}$. Then we have

$$\begin{aligned} x(n+2) + bx(n+1) + cx(n) &= c_1x_1(n+2) + c_2x_2(n+2) + x_{\mathrm{p}}(n+2) \\ &\quad + b\big(c_1x_1(n+1) + c_2x_2(n+1) + x_{\mathrm{p}}(n+1)\big) \end{aligned}$$

$$
\begin{aligned}
&\quad + c\big(c_1x_1(n) + c_2x_2(n) + x_{\mathrm{p}}(n)\big)\\
&= c_1\big(x_1(n+2) + bx_1(n+1) + cx_1(n)\big)\\
&\quad + c_2\big(x_2(n+2) + bx_2(n+1) + cx_2(n)\big)\\
&\quad + x_{\mathrm{p}}(n+2) + bx_{\mathrm{p}}(n+1) + cx_{\mathrm{p}}(n)\\
&= c_1 0 + c_2 0 + f(n) = f(n).
\end{aligned}
$$

Thus all sequences of the form (5.25) are solutions to (5.24).

Now let y be a solution to (5.24) and let $u = y - x_{\mathrm{p}}$. Then we have

$$
\begin{aligned}
u(n+2) + bu(n+1) + cu(n) &= y(n+2) - x_{\mathrm{p}}(n+2) + b\big(y(n+1) - x_{\mathrm{p}}(n+1)\big)\\
&\quad + c\big(y(n) - x_{\mathrm{p}}(n)\big)\\
&= y(n+2) + by(n+1) + cy(n)\\
&\quad - \big(x_{\mathrm{p}}(n+2) + bx_{\mathrm{p}}(n+1) + cx_{\mathrm{p}}(n)\big)\\
&= f(n) - f(n) = 0.
\end{aligned}
$$

Thus u is a solution to the corresponding homogeneous equation. It follows from Theorem 5.11 that there exist $c_1, c_2 \in \mathbf{R}$, such that $u = c_1x_1 + c_2x_2$, or $y = c_1x_1 + c_2x_2 + x_{\mathrm{p}}$. □

As a consequence of the above result we are left with the problem of finding a particular solution to a given inhomogeneous equation. There are no completely general methods, and, in general, the solution cannot be found in closed form. There are some techniques available, and we will present some of them. One of them is based on a simple idea. One tries to guess a solution. More precisely, if the right hand side is in the form of a linear combination of functions of the form

$$
r^n, \quad r^n \cos(an), \quad \text{or} \quad r^n \sin(an),
$$

then the method may succeed. Here r and a are constants, inferred from the given right hand side. We will start with some examples to clarify the method.

Example 5.14. We will find the complete solution to the equation

$$
x(n+2) + 2x(n+1) - 3x(n) = 4 \cdot 2^n.
$$

We first solve the corresponding homogeneous equation

$$
x(n+2) + 2x(n+1) - 3x(n) = 0.
$$

The characteristic equation is $r_2 + 2r - 3 = 0$ with solutions $r_1 = 1$ and $r_2 = -3$. Thus the complete solution is

$$
y(n) = c_1 + c_2(-3)^n, \quad c_1, c_2 \in \mathbf{R}.
$$

To find one solution to the inhomogeneous equation we use the guess $u(n) = c2^n$. We insert into the equation to determine c. We get

$$
c2^{n+2} + 2c2^{n+1} - 3c2^n = 4 \cdot 2^n.
$$

This leads to $c2^2 + 2c2^1 - 3c = 4$ or $c = \frac{4}{5}$. Thus a particular solution is $y_{\mathrm{p}}(n) = \frac{4}{5}2^n$. The complete solution is then

$$
x(n) = c_1 + c_2(-3)^n + \frac{4}{5}2^n, \quad c_1, c_2 \in \mathbf{R}.
$$

Example 5.15. We will find the complete solution to the equation

$$x(n+2)+4x(n)=\cos(2n). \tag{5.26}$$

We first solve the corresponding homogeneous equation

$$x(n+2)+4x(n)=0.$$

The characteristic equation is $r_2+4=0$, with solutions $r_\pm=\pm i2$. We use Theorem 5.7(iii). We have $\rho=|r_+|=2$ and $\theta=\pi/2$. Thus the solution to the homogeneous equation is

$$y(n)=c_1 2^n\cos(\tfrac{\pi}{2}n)+c_2 2^n\sin(\tfrac{\pi}{2}n).$$

If we try to find a particular solution of the form $u(n)=c\cos(2n)$, we find after substitution into the equation a term containing $\sin(2n)$. Thus the right form is $u(n)=c\cos(2n)+d\sin(2n)$. We insert this expression into the left hand side of (5.26), and then use the addition formulas to get the following result.

$$\begin{aligned}u(n+2)+4u(n)&=c\cos(2(n+2))+d\sin(2(n+2))+4(c\cos(2n)+d\sin(2n))\\&=c(\cos(2n)\cos(4)-\sin(2n)\sin(4))\\&\quad+d(\sin(2n)\cos(4)+\cos(2n)\sin(4))\\&\quad+4(c\cos(2n)+d\sin(2n))\\&=(c\cos(4)+d\sin(4)+4c)\cos(2n)\\&\quad+(-c\sin(4)+d\cos(4)+4d)\sin(2n)\end{aligned}$$

Thus to solve (5.26) we have to determine c and d, such that

$$(c\cos(4)+d\sin(4)+4c)\cos(2n)+(-c\sin(4)+d\cos(4)+4d)\sin(2n)=\cos(2n)$$

for all $n\in\mathbf{N}_0$. We now use that the sequences $\cos(2n)$ and $\sin(2n)$ are linearly independent. Thus we get the linear system of equations

$$\begin{aligned}c(4+\cos(4))+d\sin(4)&=1,\\c(-\sin(4))+d(4+\cos(4))&=0.\end{aligned}$$

The solution is

$$c=\frac{4+\cos(4)}{17+8\cos(4)},\quad d=\frac{\sin(4)}{17+8\cos(4)}.$$

Thus the complete solution to (5.26) is given by

$$x(n)=c_1 2^n\cos(\tfrac{\pi}{2}n)+c_2 2^n\sin(\tfrac{\pi}{2}n)+\frac{4+\cos(4)}{17+8\cos(4)}\cos(2n)+\frac{\sin(4)}{17+8\cos(4)}\sin(2n).$$

Example 5.16. There is a different way to find a particular solution to (5.26), based on computations with complex numbers. We note that $\cos(2n)=\operatorname{Re}e^{i2n}$. We find a particular solution to the equation

$$y(n+2)+4y(n)=e^{i2n}.$$

The particular solution to (5.26) is then found as the real part of this solution.

We note that $e^{i2n}=(e^{2i})^n$. Thus using the same technique as in Example 5.14 we guess that the solution is of the form $y(n)=ce^{2in}$, where now c can be a complex constant. Insertion gives

$$y(n+2)+4y(n)=ce^{i(2n+4)}+4ce^{i2n}$$

$$= c(e^{4i}+4)e^{2in} = e^{2in}.$$

Thus we must have

$$c = \frac{1}{e^{4i}+4} = \frac{e^{-4i}+4}{(e^{4i}+4)(e^{-4i}+4)} = \frac{e^{-4i}+4}{17+8\cos(4)}.$$

Thus the particular solution to (5.26) is given by

$$y_{\mathrm{p}}(n) = \operatorname{Re}\frac{(e^{-4i}+4)e^{2in}}{17+4\cos(4)} = \frac{4+\cos(4)}{17+8\cos(4)}\cos(2n) + \frac{\sin(4)}{17+8\cos(4)}\sin(2n).$$

This result is the same as the one in the previous example.

Example 5.17. We will find the complete solution to the equation

$$x(n+2) - x(n+1) - 6x(n) = 36n.$$

The characteristic equation is $r^2 - r - 6 = 0$ with solutions $r_1 = -2$ and $r_2 = 3$. To find a particular solution we use the guess $u(n) = d_0 + d_1 n$. Insert into the left hand side of the equation and compute as follows.

$$\begin{aligned} u(n+2) - u(n+1) - 6u(n) &= d_0 + d_1(n+2) - (d_0 + d_1(n+1)) - 6(d_0 + d_1 n) \\ &= -6d_1 n + (d_1 - 6d_0) = 36n. \end{aligned}$$

Since the sequences $\{1\}$ and $\{n\}$ are linearly independent, we get the linear system of equations $d_1 - 6d_0 = 0$ and $-6d_1 = 36$, with the solutions $d_0 = -1$ and $d_1 = -6$. Thus we have found the particular solution $u(n) = -1 - 6n$. The complete solution is then

$$x(n) = c_1(-2)^n + c_2 3^n - 1 - 6n.$$

The method used in the examples above is called the *method of undetermined coefficients*. As is evident from the second example, even simple right hand sides can lead to rather complicated particular solutions. To give a general prescription for the use of the method is rather complicated. We give a simplified description here.

Method of undetermined coefficients The method is applied to an inhomogeneous equation (5.24). There are four steps in the method:

1. Find the complete solution to the corresponding homogeneous equation in the form $x = c_1x_1 + c_2x_2$, where x_1 and x_2 are linearly independent solutions.

2. Verify that the functions x_1, x_2,and f are linearly independent (this can be done by computing their Casoratian, or sometimes seen by inspection). If they are linearly dependent, this version of the method does not apply.

3. Verify that the right hand side is a linear combination of the functions in the left hand column of Table 5.1. If this is not the case, the method cannot be applied.

4. Use the form of the solution given in the second column of Table 5.1, insert in the inhomogeneous equation (5.24), and determine the coefficients, as in the examples.

$f(n)$	form of y_{p}
r^n	cr^n
n^ν, ν an integer	$d_0 + d_1 n + \cdots + d_\nu n^\nu$
$r^n \cos(an)$	$cr^n \cos(an) + dr^n \sin(an)$
$r^n \sin(an)$	$cr^n \cos(an) + dr^n \sin(an)$

Table 5.1: Method of undetermined coefficients

In the case, where x_1, x_2, f are linearly *dependent,* and f is a linear combination of the form of functions in Table 5.1, the particular solution from this table is multiplied by n. As an example, if we instead of (5.26) consider

$$x(n+2) + 4x(n) = 2^n \sin(\tfrac{\pi}{2} n),$$

then the particular solution is of the form

$$cn2^n \cos(\tfrac{\pi}{2} n) + dn2^n \sin(\tfrac{\pi}{2} n),$$

or, alternatively, of the form

$$\operatorname{Im}(cn(2i)^n),$$

where in the second case c may be a complex constant. One finds in both cases the particular solution

$$x_{\mathrm{p}}(n) = -\frac{n}{4} \sin(\tfrac{\pi}{2} n).$$

5.3 The variable coefficient case: Homogeneous equation

We now briefly look at the general homogeneous second order difference equation (5.2). As already stated, there is no general method for solving this equation. However, we can prove a general existence and uniqueness theorem.

Theorem 5.18. *Let $b(n)$ and $c(n)$, $n \in \mathbf{N}_0$ be real sequences. Let*

$$x(n+2) + b(n)x(n+1) + c(n)x(n) = 0, \quad n \in \mathbf{N}_0. \tag{5.27}$$

Then there exist two linearly independent solutions x_1 and x_2 to (5.27)*. Let x be any solution to* (5.27)*. Then there exist $c_1, c_2 \in \mathbf{R}$, such that $x = c_1 x_1 + c_2 x_2$. Furthermore, a solution to* (5.27) *is uniquely determined by its initial values $x(0) = y0$ and $x(1) = y1$.*

Proof. We define a sequence x_1 as follows. Let $x_1(0) = 1$ and $x_1(1) = 0$. Then use (5.27) to determine $x_1(2) = -b(0)x_1(1) - c(0)x_1(0) = -c(0)$, and then $x_1(3) = -b(1)x_1(2) - c(1)x_1(1) = b(1)c(0)$. In general, we determine $x_1(n)$, $n \geq 2$, from $x_1(n-1)$ and $x_1(n-2)$. Thus we get a solution x_1 to (5.27). A second solution x_2 is determined by letting $x_2(0) = 0$ and $x_2(1) = 1$, and then repeating the arguments above. Now we use Proposition 5.4 to show that the solutions x_1 and x_2 are linearly independent. We have

$$W(0) = \begin{vmatrix} x_1(0) & x_2(0) \\ x_1(1) & x_2(1) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

which proves the claim.

Now we prove the last statement in the theorem. Let u and v be solutions to (5.27), satisfying $u(0) = v(0) = y0\ u(1) = v(1) = y1$. Let $z = u - v$. Then $z(0) = 0$ and $z(1) = 0$, and (5.27) implies that $z(n) = 0$ for all $n \in \mathbf{N}_0$, such that $u = v$, as claimed. Finally, if x is any solution to (5.27), then $x = x(0)x_1 + x(1)x_2$, by this uniqueness result. $\square$

Sometimes one can guess one solution to (5.27). Then one can use the *reduction of order method* to find a second, linearly independent, solution. We state the result in the following theorem.

Theorem 5.19 (Reduction of order)**.** *Let x_1 be a solution to* (5.27) *satisfying $x_1(n) \neq 0$ for all $n \in \mathbf{N}_0$. Then a second solution x_2 can be found by the following method. Let v be the solution to the first order homogeneous difference equation*

$$v(n+1) + \Big(1 + b(n)\frac{x_1(n+1)}{x_1(n+2)}\Big)v(n) = 0, \quad v(0) = 1. \tag{5.28}$$

and let u be a solution to the first order inhomogeneous difference equation

$$u(n+1) - u(n) = v(n). \tag{5.29}$$

Let $x_2(n) = u(n)x_1(n)$. Then x_2 is a solution to (5.27)*, and x_1, x_2 are linearly independent.*

Proof. Let u be a sequence, and let $v = \Delta u$. Let $y(n) = u(n)x_1(n)$. Repeating the computations in (5.14), one finds immediately that in order for y to solve (5.27), y must be a solution to the equation in (5.28). We take the solution v, which satisfies the initial condition in (5.28). The existence and uniqueness of this solution follows from Theorem 3.2. Then we solve (5.29), using again Theorem 3.2, and define $x_2(n) = u(n)x_1(n)$. It remains to verify that the two solutions are linearly independent. We compute their Casoratian at zero.

$$\begin{aligned} W(0) &= \begin{vmatrix} x_1(0) & x_2(0) \\ x_1(1) & x_2(1) \end{vmatrix} = \begin{vmatrix} x_1(0) & u(0)x_1(0) \\ x_1(1) & u(1)x_1(1) \end{vmatrix} \\ &= x_1(0)x_1(1)(u(1) - u(0)) = x_1(0)x_1(1)v(0). \end{aligned}$$

By assumption $x_1(0) \neq 0$ and $x_1(1) \neq 0$, and furthermore $v(0) = 1$. Thus x_1 and x_2 are linearly independent. □

5.4 The variable coefficient case: Inhomogeneous equation

We consider the inhomogeneous equation

$$x(n+2) + b(n)x(n+1) + c(n)x(n) = g(n), \quad n \in \mathbf{N}_0. \tag{5.30}$$

We need to determine one solution to this equation, which we again call a particular solution. First we note that Theorem 5.13 is valid also in the variable coefficient case. The verification is left as an exercise.

We have the following general result. The method used is called variation of parameters.

Theorem 5.20 (Variation of parameters)**.** *Assume that $c(n) \neq 0$ for all $n \in \mathbf{N}_0$. Assume that x_1 and x_2 are two linearly independent solutions to the homogeneous equation* (5.27)*. Then a particular solution to* (5.30) *is given by*

$$x_{\mathrm{p}}(n) = u_1(n)x_1(n) + u_2(n)x_2(n), \quad n \in \mathbf{N}_0,$$

where u_1 and u_2 are given by

$$u_1(n) = -\sum_{k=0}^{n-1} \frac{g(k)x_2(k+1)}{W(k+1)}, \tag{5.31}$$

$$u_2(n) = \sum_{k=0}^{n-1} \frac{g(k)x_1(k+1)}{W(k+1)}. \tag{5.32}$$

Here $W(n)$ denotes the Casoratian of x_1 and x_2.

Proof. We define

$$y(n) = u_1(n)x_1(n) + u_2(n)x_2(n)$$

and compute

$$\begin{aligned} y(n+1) = {} & u_1(n)x_1(n+1) + u_2(n)x_2(n+1) \\ & + (\Delta u_1)(n)x_1(n+1) + (\Delta u_2)(n)x_2(n+1). \end{aligned}$$

We impose the condition

$$(\Delta u_1)(n)x_1(n+1) + (\Delta u_2)(n)x_2(n+1) = 0. \tag{5.33}$$

Using this condition we compute once more

$$\begin{aligned} y(n+2) = {} & u_1(n)x_1(n+2) + u_2(n)x_2(n+2) \\ & + (\Delta u_1)(n)x_1(n+2) + (\Delta u_2)(n)x_2(n+2). \end{aligned}$$

Now insert the expressions for $y(n)$, $y(n+1)$, and $y(n+2)$ in (5.30) and simplify, using the fact that both x_1 and x_2 satisfy the homogeneous equation. This leads to the equation

$$(\Delta u_1)(n)x_1(n+2) + (\Delta u_2)(n)x_2(n+2) = g(n). \tag{5.34}$$

For each $n \in \mathbf{N}_0$ we can view the equations (5.33) and (5.34) as a pair of linear equations to determine $(\Delta u_1)(n)$ and $(\Delta u_2)(n)$. Explicitly, we have

$$(\Delta u_1)(n)x_1(n+1) + (\Delta u_2)(n)x_2(n+1) = 0. \tag{5.35}$$

$$(\Delta u_1)(n)x_1(n+2) + (\Delta u_2)(n)x_2(n+2) = g(n). \tag{5.36}$$

The determinant of the coefficient matrix is

$$\begin{vmatrix} x_1(n+1) & x_2(n+1) \\ x_1(n+2) & x_2(n+2) \end{vmatrix} = W(n+1),$$

where $W(n)$ is the Casoratian of x_1 and x_2. We use the assumption that $c(n) \neq 0$ for all $n \in \mathbf{N}_0$, the linear independence of x_1, x_2, and Lemma 5.5 to get that $W(n) \neq 0$ for all $n \in \mathbf{N}_0$. Thus we have a unique solution to the linear system. We use Cramer's method (see [3]) to solve the system. The result is

$$(\Delta u_1)(n) = \frac{\begin{vmatrix} 0 & x_2(n+1) \\ g(n) & x_2(n+2) \end{vmatrix}}{\begin{vmatrix} x_1(n+1) & x_2(n+1) \\ x_1(n+2) & x_2(n+2) \end{vmatrix}} = -\frac{x_2(n+1)g(n)}{W(n+1)}, \tag{5.37}$$

$$(\Delta u_2)(n) = \frac{\begin{vmatrix} x_1(n+1) & 0 \\ x_1(n+2) & g(n) \end{vmatrix}}{\begin{vmatrix} x_1(n+1) & x_2(n+1) \\ x_1(n+2) & x_2(n+2) \end{vmatrix}} = \frac{x_1(n+1)g(n)}{W(n+1)}. \tag{5.38}$$

Solving the two difference equations yields the expressions for u_1 and u_2 in the theorem. Let us verify that u_1 given by (5.31). We have

$$
\begin{aligned}
(\Delta u_1)(n) &= u_1(n+1) - u_1(n) \\
&= \Big(-\sum_{k=0}^{n} \frac{g(k)x_2(k+1)}{W(k+1)}\Big) - \Big(-\sum_{k=0}^{n-1} \frac{g(k)x_2(k+1)}{W(k+1)}\Big) \\
&= -\frac{g(n)x_2(n+1)}{W(n+1)}.
\end{aligned}
$$

We can also use Theorem 3.2 with the initial condition $y_0 = 0$ to get the same solution. □

5.5 Second order difference equations: Linear algebra

In this section we connect the results obtained in the previous sections with results from linear algebra. We refer to [3] for the results that we use. First, we recall from Section 4 that $\mathsf{S}(\mathbf{N}_0)$ denotes all real functions $x\colon \mathbf{N}_0 \to \mathbf{R}$, or equivalently, all real sequences indexed by $\mathbf{N}_0$. It is a real vector space, as stated in Proposition 4.1.

We now formulate some of the results above in the language of linear algebra. We limit the statements to the results in the case of a second order difference equation with constant coefficients.

Thus we consider the inhomogeneous equation

$$x(n+2) + bx(n+1) + cx(n) = f(n), \quad n \in \mathbf{N}_0, \tag{5.39}$$

and the corresponding homogeneous equation

$$x(n+2) + bx(n+1) + cx(n) = 0, \quad n \in \mathbf{N}_0. \tag{5.40}$$

We define the operator

$$L(x)(n) = x(n+2) + bx(n+1) + cx(n), \quad n \in \mathbf{N}_0, \quad x \in \mathsf{S}(\mathbf{N}_0). \tag{5.41}$$

Proposition 5.21. *The operator L defined in* (5.41) *is a linear operator from* $\mathsf{S}(\mathbf{N}_0)$ *to* $\mathsf{S}(\mathbf{N}_0)$.

Proof. The proof is the same as the proof of the superposition principle, Theorem 5.8. Here are the details. Let $x_1, x_2 \in \mathsf{S}(\mathbf{N}_0)$ and $c_1, c_2 \in \mathbf{R}$. Then we have

$$
\begin{aligned}
(L(c_1x_1 + c_2x_2))(n) &= (c_1x_1 + c_2x_2)(n+2) + b(c_1x_1 + c_2x_2)(n+1) \\
&\quad + c(c_1x_1 + c_2x_2)(n) \\
&= c_1x_1(n+2) + c_2x_2(n+2) + bc_1x_1(n+1) + bc_2x_2(n+1) \\
&\quad + cc_1x_1(n) + cc_2x_2(n) \\
&= c_1\big(x_1(n+2) + bx_1(n+1) + cx_1(n)\big) \\
&\quad + c_2\big(x_2(n+2) + bx_2(n+1) + cx_2(n)\big) \\
&= c_1L(x_1)(n) + c_2L(x_2)(n).
\end{aligned}
$$

This equality is valid for all $n \in \mathbf{N}_0$. Thus we have shown that

$$L(c_1x_1 + c_2x_2) = c_1L(x_1) + c_2L(x_2),$$

which is linearity of L. □

Based on this result we can reformulate the problem of solving the inhomogeneous equation (5.39) as follows. Given $f \in \mathsf{S}(\mathbf{N}_0)$, find $x \in \mathsf{S}(\mathbf{N}_0)$ satisfying $L(x) = f$. We state the following two results. We recall that the null space of the linear operator L is defined as

$$\ker L = \{x \in \mathsf{S}(\mathbf{N}_0) \mid L(x) = 0\}. \tag{5.42}$$

Theorem 5.22. *The linear operator $L\colon \mathsf{S}(\mathbf{N}_0) \to \mathsf{S}(\mathbf{N}_0)$ has the following two properties.*

(i) *The operator L maps $\mathsf{S}(\mathbf{N}_0)$ onto $\mathsf{S}(\mathbf{N}_0)$.*

(ii) *We have* $\dim\ker L = 2$.

Proof. To prove part (i), let $f \in \mathsf{S}(\mathbf{N}_0)$. We must find $x \in \mathsf{S}(\mathbf{N}_0)$, such that $L(x) = f$. Go back to the difference equation and write it as

$$x(n+2) = -bx(n+1) - cx(n) + f(n), \quad n \in \mathbf{N}_0.$$

We look for a solution x, which satisfies $x(0) = 0$, $x(1) = 0$. Using the equation, we find $x(2) = f(2)$, $x(3) = -bf(2) + f(3)$, etc. More formally, we prove by induction that $x(n)$ is defined and satisfies the equation for all n. This proves part (i). Note that in the proof the choice $x(0) = 0$ and $x(1) = 0$ is arbitrary. Any other choice would also lead to a proof of existence.

Concerning part (ii), then the equation $L(x) = 0$ is the homogeneous equation (5.40) written in linear algebra terms. The dimension statement is then a reformulation of Theorems 5.7 and 5.11. □

Exercises

Exercise 5.1. Fill in the details in the proof of Theorem 5.20

6 Higher order linear difference equations

In this section we give a short introduction to the theory of higher order linear difference equation. A *difference equation of order k* has the following structure

$$x(n+k) + b_{k-1}(n)x(n+k-1) + b_{k-2}(n)x(n+k-2) + \cdots + b_1(n)x(n+1) + b_0(n)x(n) = f(n), \quad n \in \mathbf{N}_0. \tag{6.1}$$

Here $b_{k-1}(n), b_{k-2}(n), \dots, b_0(n)$ are given sequences. The right hand side $f(n)$ is also a given sequence. We only consider the case of real coefficients and right hand side. The terminology is the same as in the case of the second order equations. If $f(n) = 0$ for all $n \in \mathbf{N}_0$, then the equation is said to be *homogeneous*, otherwise it is inhomogeneous.

Several results on the second order equations are valid also for higher order equations, with proofs that are essentially the same.

Theorem 6.1 (Superposition principle)**.** *Let $x_1, x_2, \dots, x_N$ be solutions to the homogeneous difference equation of order k,*

$$x(n+k) + b_{k-1}(n)x(n+k-1) + b_{k-2}(n)x(n+k-2) + \cdots + b_1(n)x(n+1) + b_0(n)x(n) = 0, \quad n \in \mathbf{N}_0. \tag{6.2}$$

Let $c_1, c_2, \dots, c_N \in \mathbf{R}$. Then $y = c_1x_1 + c_2x_2 + \cdots + c_Nx_N$ is a solution to (6.2).

A solution to (6.1) in the case $f \neq 0$ is called a *particular solution.*

Theorem 6.2. *Let $f \neq 0$ and let x_{p} be a particular solution to* (6.1). *Then the complete solution to* (6.1) *can be written as*

$$x = x_{\mathrm{h}} + x_{\mathrm{p}}, \tag{6.3}$$

where x_{h} is any solution to the homogeneous equation (6.2).

It follows from this last result that in order to find all solutions to an equation (6.1) we must solve two problem. One is to find a particular solution to the inhomogeneous equation, and the other is to find the complete solution to the corresponding homogeneous equation (6.2). In the general case this is quite complicated. We will here limit ourselves to considering the case where $b_{k-1}(n), \dots, b_0(n)$ are constant.

Thus we consider now the constant coefficient homogeneous difference equation of order k,

$$\begin{aligned} x(n+k) + b_{k-1}x(n+k-1) + b_{k-2}x(n+k-2) + \\ \cdots + b_1 x(n+1) + b_0 x(n) = 0, \quad n \in \mathbf{N}_0, \end{aligned} \tag{6.4}$$

where $b_{k-1}, \dots, b_0 \in \mathbf{R}$. The technique used to find a solution to (6.4) is the same as in the order two case. We guess a solution of the form $y(n) = r^n$, $r \neq 0$, insert in the equation to get

$$r^{n+k} + b_{k-1}r^{n+k-1} + \cdots + b_1 r^{n+1} + b_0 r^n = 0.$$

Cancelling the common non-zero factor r^n we get a polynomial equation of degree k,

$$r^k + b_{k-1}r^{k-1} + \cdots + b_1 r^1 + b_0 = 0.$$

Thus we need to know the structure of the roots of a polynomial. We state the result here. The proof will be given in another course.

Theorem 6.3. *Let*

$$r^k + b_{k-1}r^{k-1} + \cdots + b_1 r^1 + b_0$$

be a polynomial of degree k with real coefficients $b_{k-1}, \dots, b_0$. Then there exist integers $K \geq 0$ and $L \geq 0$, real numbers $\kappa_1, \dots, \kappa_K$, $\kappa_j \neq \kappa_{j'}$, $j \neq j'$, nonnegative integers $k_1, k_2, \dots, k_K$, complex numbers $\zeta_1, \dots, \zeta_L$, $\zeta_j \neq \zeta_{j'}$, $j \neq j'$, with $\operatorname{Im}\zeta_1 \neq 0, \dots, \operatorname{Im}\zeta_L \neq 0$*, and nonnegative integers $l_1, l_2, \dots, l_L$, such that*

$$\begin{aligned} & r^k + b_{k-1}r^{k-1} + \cdots + b_1 r^1 + b_0 \\ & \quad = (r-\kappa_1)^{k_1} \cdots (r-\kappa_K)^{k_K} (r-\zeta_1)^{l_1} (r-\overline{\zeta_1})^{l_1} \cdots (r-\zeta_L)^{l_L} (r-\overline{\zeta_L})^{l_L}. \end{aligned} \tag{6.5}$$

Thus the κ_j, $\zeta_{j'}$, and $\overline{\zeta}_{j'}$ are the distinct zeroes of the polynomial. The integers k_j and $2l_{j'}$ are called the multiplicities of the zeroes.

We have

$$\sum_{j=1}^{K} k_j + 2\sum_{j'=1}^{L} l_{j'} = k. \tag{6.6}$$

The general statement above is rather complicated. We will give a number of examples to clarify the statement. Consider first the polynomial of degree two $r^2 + b_1 r + b_0$. In the case $b_1^2 - 4b_0 > 0$ there are two distinct real roots κ_1 and κ_2, given by

$$\kappa_1 = \frac{-b_1 - \sqrt{b_1^2 - 4b_0}}{2}, \quad \kappa_2 = \frac{-b_1 + \sqrt{b_1^2 - 4b_0}}{2},$$

and the factorization in (6.5) takes the form

$$r^2 + b_1 r + b_0 = (r - \kappa_1)(r - \kappa_2).$$

In this case $k_1 = 1$ and $k_2 = 1$.

In the case $b_1^2 - 4b_0 = 0$ there is a real double root $\kappa_1 = -b_1/2$ and the factorization in (6.5) takes the form

$$r^2 + b_1 r + b_0 = (r - \kappa_1)^2.$$

In this case $k_1 = 2$.

Finally, in the case $b_1^2 - 4b_0 < 0$ one of the complex roots is given by

$$\zeta_1 = \frac{-b_1 + i\sqrt{4b_0 - b_1^2}}{2}.$$

The other complex root is the complex conjugate of ζ_1. The factorization (6.5) now takes the form

$$r^2 + b_1 r + b_0 = (r - \zeta_1)(r - \overline{\zeta_1}).$$

In this case $l_1 = 1$.

These examples also exemplifies the notational convention used in the statement of Theorem 6.3. In the first two cases there are no complex roots. In this case one has $L = 0$ in the statement of the theorem, and there are no complex roots. Analogously, in the third case, there are no purely real roots, hence $K = 0$ in the statement of the theorem.

For polynomials of degree three there is a general formula for the roots. However, it is very complicated, and is rarely used. For polynomials of degree four there are also formulas, but they are even more complicated.

For polynomials of degree five of higher there is no closed formula for the roots. It can be proved that this is the case. On the other hand, the existence of the factorization (6.5) can be proved for polynomials of any degree.

We now continue with the examples. For polynomials of degree three or higher we can only give examples with explicit choice of the coefficients, as explained above. One can show that one has

$$r^3 - 7r^2 + 14r - 8 = (r-1)(r-2)(r-4). \tag{6.7}$$

In this case $K = 3$, $\kappa_1 = 1$, $k_1 = 1$, $\kappa_2 = 2$, $k_2 = 1$, and $\kappa_3 = 4$, $k_3 = 1$. By convention $L = 0$. Next we consider

$$r^3 + r^2 - 21r - 45 = (r+3)^2(r-5). \tag{6.8}$$

In this case $K = 2$, $\kappa_1 = -3$, $k_1 = 2$ and $\kappa_2 = 5$, $k_2 = 1$. By convention $L = 0$.

As the next example of polynomials of degree three we consider

$$r^3 - 12r^2 + 22r - 20 = (r-10)(r-1-i)(r-1+i). \tag{6.9}$$

In this case $K = 1$, $\kappa_1 = 10$, $k_1 = 1$ and $L = 1$, $\zeta_1 = 1 + i$, $l_1 = 1$.

As the final example of polynomials of degree three we consider

$$r^3 + 9r^2 + 27r + 27 = (r+3)^3 . \tag{6.10}$$

In this case $K = 3$, $\kappa_1 = -3$, $k_1 = 3$. By convention $L = 0$.

The four examples of polynomials of degree three cover all the cases that may occur. In (6.7) we have three real distinct roots, each with multiplicity one. In (6.8) we have two real distinct roots, one with multiplicity two, and one with multiplicity one. In (6.9) we have one real root, with multiplicity one, and a pair of complex conjugate roots. Finally, in (6.10) we have one real root of multiplicity three.

Concerning polynomials of degree four we give just one example. We consider

$$r^4 - 2r^3 + 6r^2 - 2r + 5 = (r-1-2i)(r-1+2i)(r+i)(r-i) . \tag{6.11}$$

In this case $K = 0$, $L = 2$, $\zeta_1 = 1 + 2i$, $l_1 = 1$, and $\zeta_2 = i$, $l + 2 = 1$.

Now we state a result for constant coefficient homogeneous difference equations of order three, based on the general result Theorem 6.3.

Theorem 6.4. *For a constant coefficient homogeneous difference equations of order three,*

$$x(n+3) + b_2 x(n+2) + b_1 x(n+1) + b_0 x(n) = 0, \quad n \in \mathbf{N}_0, \tag{6.12}$$

with $b_2, b_1, b_0 \in \mathbf{R}$ we have the following results. Let $p(r) = r^3 + b_2 r^2 + b_1 r + b_0$ denote the characteristic polynomial.

(i) *Assume that $p(r)$ has three distinct real roots κ_1, κ_2, and κ_3. Define*

$$x_j(n) = \kappa_j^n, \quad n \in \mathbf{N}_0, \quad j = 1,2,3. \tag{6.13}$$

Then x_j, $j = 1,2,3$, are three linearly independent solutions to (6.12).

(ii) *Assume that $p(r)$ has two distinct real roots κ_1 and κ_2, with multiplicities two and one, respectively. Define*

$$x_1(n) = \kappa_1^n, \quad x_2(n) = n\kappa_1^n, \quad x_3(n) = \kappa_2^n. \tag{6.14}$$

Then x_j, $j = 1,2,3$, are three linearly independent solutions to (6.12).

(iii) *Assume that $p(r)$ has one real root κ_1 and a pair of complex roots ζ_1, $\overline{\zeta}_1$. Let $\zeta_1 = \rho_1(\cos(\theta_1) + i\sin(\theta_1))$. Define*

$$x_1(n) = \kappa_1^n, \quad x_2(n) = \rho_1^n \cos(n\theta_1), \quad x_3(n) = \rho_1^n \sin(n\theta_1). \tag{6.15}$$

Then x_j, $j = 1,2,3$, are three linearly independent solutions to (6.12).

(iii) *Assume that $p(r)$ has one real root κ_1 of multiplicity three. Define*

$$x_1(n) = \kappa_1^n, \quad x_2(n) = n\kappa_1^n, \quad x_3(n) = n^2\kappa_1^n. \tag{6.16}$$

Then x_j, $j = 1,2,3$, are three linearly independent solutions to (6.12).

As an example, we consider the difference equation

$$x(n+3) - 7x(n+2) + 14x(n+1) - 8x(n) = 0.$$

The characteristic equation is given by (6.7). Using the factorization and the theorem we have three linearly independent solutions

$$x_1(n) = 1, \quad x_2(n) = 2^n, \quad x_3(n) = 4^n.$$

Concerning the inhomogeneous equation, then one can again use the method of undetermined coefficients. As an example we consider

$$x(n+3) - 7x(n+2) + 14x(n+1) - 8x(n) = 3^n.$$

We try $y(n) = c3^n$, which we insert into the equation. A simple computation yields that a particular solution is given by $y(n) = -\frac{1}{2}3^n$. Thus the complete solution is given by

$$x(n) = c_1 + c_2 2^n + c_3 4^n - \tfrac{1}{2}3^n.$$

A result similar to Theorem 5.11 holds for third order difference equations. This means that once we have found three linearly independent solutions to the homogeneous equation (6.12), any other solution to this equation can be written as a linear combination of these three solutions.

7 Systems of first order difference equations

We will now consider systems of first order difference equations. We start with a homogeneous system of two difference equations with constant coefficients. In analogy with the results in Section 3 we write the system as

$$x_1(n+1) = a_{11}x_1(n) + a_{12}x_2(n), \tag{7.1}$$

$$x_2(n+1) = a_{21}x_1(n) + a_{22}x_2(n). \tag{7.2}$$

The system can be written in matrix form as

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}. \tag{7.3}$$

We now introduce the notation

$$\mathbf{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{7.4}$$

such that we can write the system as a vector-matrix equation

$$\mathbf{x}(n+1) = A\mathbf{x}(n). \tag{7.5}$$

Written in this form the equation has the same form as the first order equations considered in Section 3.

The proof given in Section 3 can now be repeated and leads to the solution

$$\mathbf{x}(n) = A^n\mathbf{x}(0), \quad n \in \mathbf{N}. \tag{7.6}$$

However, one should keep in mind that A^n is a power of a matrix. By convention $A^0 = I$, the identity matrix.

The inhomogeneous system of two first order difference equations with constant coefficients is given by

$$x_1(n+1) = a_{11}x_1(n) + a_{12}x_2(n) + c_1, \tag{7.7}$$
$$x_2(n+1) = a_{21}x_1(n) + a_{22}x_2(n) + c_2. \tag{7.8}$$

We define

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Thus the inhomogeneous equation can be written in vector-matrix form as

$$\mathbf{x}(n+1) = A\mathbf{x}(n) + \mathbf{c}. \tag{7.9}$$

The arguments leading to the solution formula (3.5) can be repeated. However, we must take care of writing terms in the right order, in contrast to (3.5). The result is

$$\mathbf{x}(n) = A^n\mathbf{x}(0) + \sum_{k=0}^{n-1} A^k\mathbf{c}, \quad n \in \mathbf{N}. \tag{7.10}$$

The results for more than two equations are almost the same. For example, three equations are given as

$$x_1(n+1) = a_{11}x_1(n) + a_{12}x_2(n) + a_{13}x_3(n) + c_1, \tag{7.11}$$
$$x_2(n+1) = a_{21}x_1(n) + a_{22}x_2(n) + a_{23}x_3(n) + c_2, \tag{7.12}$$
$$x_3(n+1) = a_{31}x_1(n) + a_{32}x_2(n) + a_{33}x_3(n) + c_3. \tag{7.13}$$

We state the general case in the following theorem.

Theorem 7.1. *Let $k \in \mathbf{N}$, $k \geq 2$. A linear inhomogeneous system of k first order difference equations is given by*

$$x_1(n+1) = a_{11}x_1(n) + a_{12}x_2(n) + \cdots + a_{1k}x_k(n) + c_1, \tag{7.14}$$
$$x_2(n+1) = a_{21}x_1(n) + a_{22}x_2(n) + \cdots + a_{2k}x_k(n) + c_2, \tag{7.15}$$
$$\vdots \tag{7.16}$$
$$x_k(n+1) = a_{k1}x_1(n) + a_{k2}x_2(n) + \cdots + a_{kk}x_k(n) + c_k. \tag{7.17}$$

Using the vector and matrix notation

$$\mathbf{x}(n) = \begin{bmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_k(n) \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1k} \\ a_{21} & a_{22} & \cdots & a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kk} \end{bmatrix}, \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}, \tag{7.18}$$

the system is written as

$$\mathbf{x}(n+1) = A\mathbf{x}(n) + \mathbf{c}. \tag{7.19}$$

The solution is given by

$$\mathbf{x}(n) = A^n\mathbf{x}(0) + \sum_{j=0}^{n-1} A^j\mathbf{c}, \quad n \in \mathbf{N}. \tag{7.20}$$

If 1 *is not an eigenvalue of the matrix A, then the solution can be written as*

$$\mathbf{x}(n) = A^n\mathbf{x}(0) + (A^n - I)(A - I)^{-1}\mathbf{c}. \tag{7.21}$$

The only part of the proof differing from the one given in Section 3 is the derivation of the formula (7.21). The result is stated here.

Lemma 7.2. *Let A be an $N \times N$ matrix. Assume that* 1 *is not an eigenvalue of A. Then we have that*

$$\sum_{j=0}^{n-1} A^j = (A^n - I)(A - I)^{-1} \quad \text{for } n \geq 1. \tag{7.22}$$

Proof. We prove the result by induction. Consider first $n = 1$. The left had side is

$$\sum_{j=0}^{1-1} A^j = A^0 = I.$$

The right hand side is

$$(A^1 - I)(A - I)^{-1} = I.$$

Thus the formula (7.22) holds for $n = 1$. Next consider an arbitrary $n \geq 1$, and assume that (7.22) holds for this n. We then consider the formula for $n + 1$ and compute as follows.

$$\begin{aligned}
\sum_{j=0}^{n} A^j &= A^n + \sum_{j=0}^{j-1} A^j \\
&= \big(A^n(A - I) + (A^n - I)\big)(A - I)^{-1} \\
&= (A^{n+1} - I)(A - I)^{-1}.
\end{aligned}$$

Thus the formula also holds for $n + 1$ and the induction argument is completed. □

Theorem 7.1 is stated with a constant inhomogeneous term. We have the following result in case $\mathbf{c}$ is a function of n.

Theorem 7.3. *Let A be a $k \times k$ matrix, and let $\mathbf{c}\colon \mathbf{N}_0 \to \mathbf{R}^k$ be a function. Then the system of first order difference equations*

$$\mathbf{x}(n + 1) = A\mathbf{x}(n) + \mathbf{c}(n) \tag{7.23}$$

has the solution

$$\mathbf{x}(n) = A^n\mathbf{x}(0) + \sum_{j=0}^{n-1} A^{n-1-j}\mathbf{c}(j). \tag{7.24}$$

The formulas given for solving the inhomogeneous case are sometimes complicated. We note that the *method of undetermined coefficients* can be applied also to systems. Let us explain how in the case of a constant inhomogeneous term. Thus we consider

$$\mathbf{x}(n + 1) = A\mathbf{x}(n) + \mathbf{c}. \tag{7.25}$$

We have the same structure as in all the other cases of linear inhomogeneous equations, see also Theorem 6.2. The complete solution to (7.25) is given as

$$x(n) = x_{\mathrm{h}}(n) + x_{\mathrm{p}}(n), \tag{7.26}$$

where $x_{\mathrm{h}}(n)$ is the complete solution to the corresponding homogeneous equation, and x_{p} is a particular solution to the inhomogeneous equation.

In order to apply this result we guess that the solution to (7.25) is a constant

$$\mathbf{y}(n) = \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

Inserting into the equation yields the equation

$$\mathbf{y} = A\mathbf{y} + \mathbf{c} \quad \text{or} \quad (I - A)\mathbf{y} = \mathbf{c}$$

If 1 is not an eigenvalue of A, we can always solve this linear system for $\mathbf{y}$. If 1 is an eigenvalue, we can use the general formula (7.20).

Example 7.4. To illustrate the results above we consider the system

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ 2 & 2 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \end{bmatrix}.$$

The matrix above is denoted by A. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 4$. We use the results presented later in this section, in Theorem 7.7. After some computations we find that

$$A^n = 3^n \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + 4^n \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}.$$

To solve the inhomogeneous problem we first use the expression (7.21). A computation shows that

$$(A - I)^{-1} = \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix}.$$

Thus without simplifications we get the complete solution

$$\begin{aligned} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} &= \left(3^n \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + 4^n \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\right) \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \\ &\quad + \left(3^n \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + 4^n \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \end{bmatrix}. \end{aligned}$$

Now we can also use the methods of undetermined coefficients. This leads to the linear system $(I - A)\mathbf{y} = \mathbf{c}$, which has the solution

$$\mathbf{y} = -\frac{2}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Thus we get the complete solution written as

$$\begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} = \left(3^n \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix} + 4^n \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix}\right) \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} - \frac{2}{3} \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Here $d_1, d_2 \in \mathbf{R}$ are arbitrary. The difference between the two approaches is that in the first case we can immediately insert $x_1(0)$ and $x_2(0)$ to find the solution satisfying a given initial condition, whereas in the second case we must solve a linear system first to determine the values of d_1 and d_2 from the initial condition.

7.1 Second order difference equation as a system

We now show how a second order difference equation can be written as a system of two first order difference equations. Using the notation of Section 5 we consider first a homogeneous equation

$$x(n+2) + bx(n+1) + cx(n) = 0. \tag{7.27}$$

Define

$$x_1(n) = x(n), \tag{7.28}$$
$$x_2(n) = x(n+1). \tag{7.29}$$

Then we can rewrite (7.27) as

$$x_1(n+1) = x_2(n), \tag{7.30}$$
$$x_2(n+1) = -bx_2(n) - cx_1(n), \tag{7.31}$$

or in matrix form

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix}. \tag{7.32}$$

A similar computation for the inhomogeneous system

$$x(n+2) + bx(n+1) + cx(n) = f(n) \tag{7.33}$$

yields the matrix equation

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \end{bmatrix} + \begin{bmatrix} 0 \\ f(n) \end{bmatrix}. \tag{7.34}$$

We can also carry out the argument in the opposite direction. Assume that we have a system of the particular form (7.34). Assume that we have found a solution $\left[\begin{smallmatrix} x_1(n) \\ x_2(n) \end{smallmatrix}\right]$ to (7.34). Then $x(n) = x_1(n)$ is a solution to (7.33).

The same computations can be carried out for a difference equation of order three. We now use the systematic notation from Section 6.

$$x(n+3) + b_2x(n+2) + b_1x(n+1) + b_0x(n) = f(n). \tag{7.35}$$

We define

$$x_1(n) = x(n), \quad x_2(n) = x(n+1), \quad x_3(n) = x(n+2), \tag{7.36}$$

which leads to the matrix equation

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \\ x_3(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -b_0 & -b_1 & -b_2 \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ f(n) \end{bmatrix}. \tag{7.37}$$

Again, the argument is also valid in the opposite direction, i.e. $x(n) = x_1(n)$ from the solution of the system is a solution to the third order difference equation.

We formulate the general case in the form of an *equivalence Theorem.*

Theorem 7.5. *Consider a difference equation of order k,*

$$x(n+k) + b_{k-1}x(n+k-1) + b_{k-2}x(n+k-2) + \\ \cdots + b_1x(n+1) + b_0x(n) = f(n). \quad (7.38)$$

Define

$$x_1(n) = x(n), \quad x_2(n) = x(n+1), \ldots, x_k(n) = x(n+k-1). \quad (7.39)$$

Then we get the matrix equation

$$\begin{bmatrix} x_1(n+1) \\ x_2(n+1) \\ \vdots \\ x_{k-1}(n+1) \\ x_k(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{k-1} \end{bmatrix} \begin{bmatrix} x_1(n) \\ x_2(n) \\ \vdots \\ x_{k-1}(n) \\ x_k(n) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ f(n) \end{bmatrix}. \quad (7.40)$$

We have the following equivalence. Any solution to the k-th order difference equation (7.38) *gives a solution to the system* (7.40) *through the definition* (7.39)*. Conversely, if* $x_1(n),\ldots,x_k(n)$ *is a set of solutions to the system* (7.40)*, then* $x(n) = x_1(n)$ *is a solution to the k-th order difference equation* (7.38)*.*

We now consider how to compute the powers A^n of a matrix, which is what is needed to get explicit solutions to the systems of first order difference equations.

We start with a system with two first order equations. It is written in matrix form in (7.5), where the 2×2 matrix A is given in (7.4). The solution is given by the powers of A, as in (7.6). First we assume that A can be *diagonalized*, see [3, p. 318]. This means that we have

$$A = PDP^{-1}, \quad D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}. \quad (7.41)$$

Here λ_1 and λ_2 are the eigenvalues of A and P is a matrix whose columns are corresponding linearly independent eigenvectors.

Now we see how (7.41) can be used to compute the powers. We have

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(PP^{-1})DP^{-1} = PD^2P^{-1}$$

and then

$$A^3 = A^2A = (PD^2P^{-1})(PDP^{-1}) = PD^2(PP^{-1})DP^{-1} = PD^3P^{-1}.$$

In general we get

$$A^n = PD^nP^{-1}. \quad (7.42)$$

Since we have

$$D^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix},$$

this gives the solution, provided we can compute P and D.

Let us now give an examples of an explicit system and its solution.

Example 7.6. We consider the system

$$x_1(n+1) = 10x_1(n) - 24x_2(n), \quad (7.43)$$

$$x_2(n) = 4x_1(n) - 10x_2(n). \quad (7.44)$$

Thus we have the matrix

$$A = \begin{bmatrix} 10 & -24 \\ 4 & -10 \end{bmatrix}.$$

This matrix can be diagonalized. The result is that

$$P = \begin{bmatrix} 3 & 2 \\ 1 & 1 \end{bmatrix}, \quad \text{with} \quad P^{-1} = \begin{bmatrix} 1 & -2 \\ -1 & 3 \end{bmatrix},$$

and

$$D = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}.$$

This allows us to find

$$A^n = 2^n \begin{bmatrix} 3 - 2(-1)^n & -6 + 6(-1)^n \\ 1 - (-1)^n & -2 + 3(-1)^n \end{bmatrix}.$$

Not every matrix can be diagonalized. An example is the matrix

$$B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}. \tag{7.45}$$

There is a general algorithm, which avoids diagonalization, and thus can be applied to all matrices. It is a discrete version of Putzer's algorithm, see [2, p. 118]. We state this algorithm for the case of 2×2 matrices.

Theorem 7.7 (Putzer's algorithm)**.** *Let A be a 2×2 matrix.*

1. *Assume that A has two different eigenvalues λ_1 and λ_2. Then we have*

$$A^n = \lambda_1^n I + \frac{\lambda_1^n - \lambda_2^n}{\lambda_1 - \lambda_2}(A - \lambda_1 I), \quad n \geq 1. \tag{7.46}$$

2. *Assume that A has only one eigenvalue λ_1. Then we have*

$$A^n = \lambda_1^n I + n\lambda_1^{n-1}(A - \lambda_1 I), \quad n \geq 1. \tag{7.47}$$

Let us apply this theorem to the matrix B defined in (7.45). We see that the only eigenvalue is 1. Thus the theorem gives

$$B^n = 1^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + n1^{n-1}\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} - 1\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

Example 7.8. We briefly explain how the application of Putzer's algorithm to the matrix from Example 7.6 leads to the same result as the one obtained in this example. Let again

$$A = \begin{bmatrix} 10 & -24 \\ 4 & -10 \end{bmatrix}.$$

The eigenvalues of A can be found using the characteristic equation

$$\det(A - \lambda I) = \lambda^2 - 4 = 0,$$

leading to $\lambda_1 = 2$ and $\lambda_2 = -2$. Insert into (7.46) to get

$$A^n = 2^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{2^n - (-2)^n}{2 - (-2)}\left(\begin{bmatrix} 10 & -24 \\ 4 & -10 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= 2^n\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + (1-(-1)^n)\frac{1}{4}\begin{bmatrix}8 & -24\\ 4 & -12\end{bmatrix}\right)$$
$$= 2^n\begin{bmatrix}3-2(-1)^n & -6+6(-1)^n\\ 1-(-1)^n & -2+3(-1)^n\end{bmatrix}.$$

Thus we get the same result as in Example 7.6.

Let us now compare the results for second order difference equations obtained in Section 5 with the results for systems in this Section. We start with the homogeneous equation. Thus we compare solutions to

$$x(n+2) + bx(n+1) + cx(n) = 0 \tag{7.48}$$

with solutions to

$$\begin{bmatrix}x_1(n+1)\\ x_2(n+1)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -c & -b\end{bmatrix}\begin{bmatrix}x_1(n)\\ x_2(n)\end{bmatrix}. \tag{7.49}$$

We recall from Section 5 that the two linearly independent solutions of (7.48) are determined by the characteristic equation (5.9), which we repeat here

$$r^2 + br + c = 0.$$

The eigenvalues of the matrix in (7.49) are determined by the roots of what is called the characteristic polynomial

$$\det(A-\lambda I) = \begin{vmatrix}-\lambda & 1\\ -c & -b-\lambda\end{vmatrix} = \lambda^2 + b\lambda + c.$$

We see that it is the same polynomial of degree two.

We consider first the case when there are two real roots, i.e. the case $b^2-4c>0$. We choose

$$\lambda_1 = \tfrac{1}{2}(-b+\sqrt{b^2-4c}), \quad \lambda_2 = \tfrac{1}{2}(-b+\sqrt{b^2-4c}).$$

We recall that $\lambda_1+\lambda_2 = -b$ and $\lambda_1\lambda_2 = c$.

It is clear from Theorem 7.5 how to get two solutions to the system from two linearly independent solutions to the second order equations. Now we show how to get the two linearly independent solution λ_1^n and λ_2^n to (7.48) from the system. We do this using Putzer's algorithm. Using the above results we note that

$$A-\lambda_1 I = \begin{bmatrix}-\lambda_1 & 1\\ -c & -b-\lambda_1\end{bmatrix} = \begin{bmatrix}-\lambda_1 & 1\\ -\lambda_1\lambda_2 & \lambda_2\end{bmatrix}.$$

The Putzer algorithm (7.46) then gives the following expression for the powers of A.

$$A^n = \begin{bmatrix}\lambda_1^n & 0\\ 0 & \lambda_1^n\end{bmatrix} + \frac{\lambda_1^n-\lambda_2^n}{\lambda_1-\lambda_2}\begin{bmatrix}-\lambda_1 & 1\\ -\lambda_1\lambda_2 & \lambda_2\end{bmatrix}.$$

Now the solution to the system is given by

$$\begin{bmatrix}x_1(n)\\ x_2(n)\end{bmatrix} = A^n\begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}$$

By choosing appropriate initial conditions we get the two solutions to the second order equation. Recall that it is the component $x_1(n)$ that yields the solution to (7.48). Taking

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix},$$

we get the solution λ_1^n. Taking

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix},$$

we get the solution λ_2^n.

The same computations work in the case $b^2 - 4c < 0$. Here we get the pair of complex conjugate solutions, and we can get the two real solutions as in Section 5.

Now we consider the case when there is only one solution λ_1, i.e. the case $b^2 - 4c = 0$. In this case we get from Putzer's algoritm the expression

$$A^n = \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_1^n \end{bmatrix} + n\lambda_1^{n-1} \begin{bmatrix} -\lambda_1 & 1 \\ -\lambda_1^2 & \lambda_1 \end{bmatrix}.$$

Taking again

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix},$$

we get the solution λ_1^n. To get the second solution we take

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda_1 \end{bmatrix},$$

which gives the solution $n\lambda_1^n$. Thus we have elaborated the connection between the second order equation (7.48) and the coorsponding system (7.49).

7.2 Further results on matrix powers

We give a few additional results concerning matrix powers. One can prove the following result, which is called the generalized spectral theorem for a 2×2 matrix.

Theorem 7.9. *Let A be a 2×2 matrix.*

(i) *Assume that A has two different eigenvalues λ_1 and λ_2. Then there exist 2×2 matrices P and Q with the following four properties*

$$P^2 = P, \tag{7.50}$$

$$Q^2 = Q, \tag{7.51}$$

$$PQ = 0, \tag{7.52}$$

$$P + Q = I, \tag{7.53}$$

such that

$$A = \lambda_1 P + \lambda_2 Q. \tag{7.54}$$

We have that

$$P\mathbf{R}^2 = \ker(A - \lambda_1 I) \quad \textit{and} \quad Q\mathbf{R}^2 = \ker(A - \lambda_2 I) \tag{7.55}$$

such that the ranges of P and Q are the one dimensional eigenspaces. Furthermore, for all $n \geq 1$ we have that

$$A^n = \lambda_1^n P + \lambda_2^n Q. \tag{7.56}$$

(ii) *Assume that A has only one distinct eigenvalue λ_1. Then there exists a 2×2 matrix N with the property*

$$N^2 = 0, \tag{7.57}$$

such that

$$A = \lambda_1 I + N. \tag{7.58}$$

Furthermore, for all $n \geq 1$ we have that

$$A^n = \lambda_1^n I + n\lambda_1^{n-1} N. \tag{7.59}$$

A matrix P satisfying (7.50) is called a projection. The matrix $Q = I - P$ is called the complementary projection. A matrix satisfying (7.57) is said to be nilpotent of order 2.

We are not going to prove this result, but we give some examples. We note that one can either prove Putzer's algorithm, and derive the results in the Theorem, or one can prove the Theorem, and derive Putzer's algorithm.

Example 7.10. Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

The eigenvalues are

$$\lambda_1 = -3, \quad \lambda_2 = 2.$$

Using Putzer's algorithm one finds after some computations

$$A^n = (-3)^n \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} + 2^n \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

The matrices

$$P = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix} \quad \text{and} \quad Q = \frac{1}{5} \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}$$

satisfy all the conditions in Theorem 7.9, as one can verify by simple computations.

Example 7.11. Consider the matrix

$$A = \begin{bmatrix} 0 & -9 \\ 1 & 6 \end{bmatrix}.$$

This matrix has only one eigenvalue

$$\lambda_1 = 3.$$

Putzer's algorithm gives the following result

$$A^n = 3^n \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + n3^{n-1} \begin{bmatrix} -3 & -9 \\ 1 & 3 \end{bmatrix}.$$

The matrix

$$N = \begin{bmatrix} -3 & -9 \\ 1 & 3 \end{bmatrix}$$

satisfies $N^2 = 0$, as a simple computation shows.

References

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