A Short Introduction to Complex Analysis

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1 Introduction

These notes are intended for use in the course on real and complex functions at Aalborg University. They start with the basic results on analytic functions, and end with a proof of a version of the theorem on residues for a meromorphic function. These notes are used in conjunction with the textbook [3], and several references will be made to this book.

2 Holomorphic functions

We start by defining holomorphic functions. These functions are simply functions of a complex variable that can be differentiated in the complex sense.

Definition 2.1. Let $G \subseteq \mathbf{C}$ be an open subset. A function $f\colon G \to \mathbf{C}$ is said to be differentiable in the complex sense at $z_0 \in G$, if

$$\lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The limit is denoted by $f'(z_0)$.

Definition 2.2. Let $G \subseteq \mathbf{C}$ be an open subset. A function $f\colon G \to \mathbf{C}$ is said to be holomorphic in G, if it is differentiable in the complex sense at all points in G. The set of holomorphic functions is denoted by $\mathcal{H}(G)$.

Let us note that the rules for differentiation of a sum, a product, and a quotient of two complex functions are the same as in the real case. The proofs given in [3, Chapter 4] are valid in the complex case.

We recall that we can decompose a complex number into its real and imaginary parts. Applying this decomposition at each value of a complex function we get a decomposition $f = u + iv$, where $u = \mathrm{Re}(f)$ and $v = \mathrm{Im}(f)$ will be viewed as functions of the real variable pair (x, y) corresponding to $z = x + iy$. We often identify the point (x, y) in $\mathbf{R}^2$ with the point $z = x + iy$ in $\mathbf{C}$. This identification should be kept in mind at various places in these notes. Thus a function from $G \subseteq \mathbf{C}$ to $\mathbf{C}$ can also be viewed as a function from a subset of $\mathbf{R}^2$ to $\mathbf{R}^2$. Thus we may write a function as $f(z)$ or $f(x, y)$, depending on whether we view it as defined on a subset of $\mathbf{C}$, or a subset of $\mathbf{R}^2$.

An open ball in $\mathbf{R}^2$ centered at 0 and with radius δ is denoted by $B(0, \delta)$. Let $G \subseteq \mathbf{R}^2$ be an open subset. We recall from [2] that a function $u\colon G \to \mathbf{R}$ is differentiable in the *real sense* (or has a total derivative) at a point $(x_0, y_0) \in G$, if and only if there exist a $\delta > 0$, a function $E\colon B(0, \delta) \to \mathbf{R}$ with $E(x, y) \to 0$ as $(x, y) \to (0, 0)$, and two real numbers a and b, such that

$$\begin{aligned} u(x, y) = u(x_0, y_0) &+ a(x - x_0) + b(y - y_0) \\ &+ \|(x - x_0, y - y_0)\| E(x - x_0, y - y_0) \end{aligned} \tag{2.1}$$

for $(x - x_0, y - y_0) \in B(0, \delta)$. In this case the partial derivatives exist at (x_0, y_0), and we have

$$\frac{\partial u}{\partial x}(x_0, y_0) = a, \quad \frac{\partial u}{\partial y}(x_0, y_0) = b. \tag{2.2}$$

This result has an immediate generalization to the case of differentiability in the complex sense.

Lemma 2.3. *A function $f\colon G \to \mathbf{C}$ is differentiable in the complex sense at $z_0 \in G$, if and only if there exist $c \in \mathbf{C}$ and $E\colon B(0, \delta) \to \mathbf{C}$ with $E(h) \to 0$ as $h \to 0$, such that*

$$f(z) = f(z_0) + c(z - z_0) + |z - z_0| E(z - z_0) \tag{2.3}$$

for $z - z_0 \in B(0, \delta)$. If f is differentiable at z_0, then $f'(z_0) = c$.

We have the following result.

Theorem 2.4. *Let $G \subseteq \mathbf{C}$ be an open subset. A function $f = u + iv$ from G to $\mathbf{C}$ is differentiable in the complex sense at the point $z_0 = x_0 + iy_0 \in G$, if and only if the functions u and v both are differentiable in the real sense at (x_0, y_0) with partial derivatives satisfying the Cauchy-Riemann equations*

$$\frac{\partial u}{\partial x}(x_0, y_0) = \frac{\partial v}{\partial y}(x_0, y_0), \quad \frac{\partial v}{\partial x}(x_0, y_0) = -\frac{\partial u}{\partial y}(x_0, y_0). \tag{2.4}$$

In this case we have

$$f'(z_0) = \frac{\partial u}{\partial x}(x_0, y_0) + i\frac{\partial v}{\partial x}(x_0, y_0). \tag{2.5}$$

Proof. Assume first that f is differentiable in the complex sense at z_0 with derivative $f'(z_0) = c = a+ib$. Then we can find a function E, such that (2.3) holds. Take the real part of this equation, with the notation $f = u + iv$ and $E = E_1 + iE_2$. Note that $|z - z_0| = \|(x - x_0, y - y_0)\|$. The result is

$$\begin{aligned} u(x, y) = u(x_0, y_0) &+ a(x - x_0) - b(y - y_0) \\ &+ \|(x - x_0, y - y_0)\| E_1(x - x_0, y - y_0). \end{aligned} \tag{2.6}$$

Thus it follows from [2] that u is differentiable in the real sense at (x_0, y_0), and that we have

$$\frac{\partial u}{\partial x}(x_0, y_0) = a, \quad \frac{\partial u}{\partial y}(x_0, y_0) = -b. \tag{2.7}$$

Analogously, taking the imaginary part of (2.3), we find that

$$\begin{aligned} v(x, y) = v(x_0, y_0) &+ b(x - x_0) + a(y - y_0) \\ &+ \|(x - x_0, y - y_0)\| E_2(x - x_0, y - y_0). \end{aligned} \tag{2.8}$$

Thus v is differentiable in the real sense at (x_0, y_0), and we have

$$\frac{\partial v}{\partial x}(x_0, y_0) = b, \quad \frac{\partial v}{\partial y}(x_0, y_0) = a. \tag{2.9}$$

Comparing (2.7) and (2.9), we see that the Cauchy-Riemann equations (2.4) hold. Since $f'(z_0) = a + ib$, it also follows that (2.5) holds.

Conversely, assume now that both u and v are differentiable in the real sense at (x_0, y_0), and furthermore that the Cauchy-Riemann equations (2.4) hold. To simplify the notation, write

$$\tilde{a} = \frac{\partial u}{\partial x}(x_0, y_0) \quad \text{and} \quad \tilde{b} = \frac{\partial u}{\partial y}(x_0, y_0).$$

and also

$$\alpha = \frac{\partial v}{\partial x}(x_0, y_0) \quad \text{and} \quad \beta = \frac{\partial v}{\partial y}(x_0, y_0).$$

Since (2.4) hold, we have $\tilde{a} = \beta$ and $\tilde{b} = -\alpha$. Furthermore, we can find functions E_1 and E_2, defined on a small ball around zero, such that (2.1) holds for u and v, with E_1 and E_2, respectively. Now we compute as follows

$$\begin{aligned} f(z) &= u(x, y) + iv(x, y) \\ &= u(x_0, y_0) + \tilde{a}(x - x_0) + \tilde{b}(y - y_0) \end{aligned}$$

$$
\begin{aligned}
&\quad + \|(x-x_0, y-y_0)\| E_1(x-x_0, y-y_0) \\
&\quad + i\big[v(x_0,y_0) + \alpha(x-x_0) + \beta(y-y_0) \\
&\qquad + \|(x-x_0, y-y_0)\| E_2(x-x_0, y-y_0)\big] \\
&= u(x_0,y_0) + iv(x_0,y_0) \\
&\quad + \tilde{a}\big((x-x_0) + i(y-y_0)\big) + i\alpha\big((x-x_0) + i(y-y_0)\big) \\
&\qquad + \|(x-x_0, y-y_0)\| \big(E_1(x-x_0, y-y_0) + iE_2(x-x_0, y-y_0)\big) \\
&= f(z_0) + (\tilde{a} + i\alpha)(z-z_0) + |z-z_0| E(z-z_0),
\end{aligned}
$$

where we have defined $E = E_1 + iE_2$ and used our notational conventions for points in $\mathbf{C}$ and $\mathbf{R}^2$. It follows from Lemma 2.3 that f is differentiable in the complex sense at z_0, and that (2.5) holds. □

The Cauchy-Riemann equations express the fact that the partial derivatives in the real sense of the real and imaginary parts of a function differentiable in the complex sense cannot be arbitrary. This fact has several important consequences. We have the following result, whose proof we will omit. It can be found in [1, Theorem 5.23].

Theorem 2.5. *Let $G \subseteq \mathbf{C}$ be an open and connected set, and let $f = u + iv \in \mathcal{H}(G)$. If any one of u, v or $|f|$ is constant on G, then f is constant on G. If $f'(z) = 0$ for all $z \in G$, then f is constant in G.*

There is a useful criterion for determining whether a given function is differentiable in the complex sense. It is obtained by combining Theorem 2.4 with [2].

Theorem 2.6. *Let $G \subseteq \mathbf{C}$ be open and $f = u + iv$ a complex valued function defined on G. Let $z_0 = x_0 + iy_0 \in G$. If the partial derivatives of u and v exist in G, are continuous at (x_0, y_0), and satisfy the Cauchy-Riemann equations* (2.4)*, then f is differentiable in the complex sense at z_0.*

Example 2.7. Let $f(z) = \exp(z)$, $z \in \mathbf{C}$. Then the decomposition $f = u + iv$ is given by

$$
u(x,y) = e^x \cos(y), \quad v(x,y) = e^x \sin(y).
$$

Clearly the partial derivatives of u and v exist and are continuous at all points in $\mathbf{R}^2$. Furthermore, we have

$$
\begin{aligned}
\frac{\partial u}{\partial x}(x,y) &= e^x \cos(y), & \frac{\partial u}{\partial y}(x,y) &= -e^x \sin(y), \\
\frac{\partial v}{\partial x}(x,y) &= e^x \sin(y), & \frac{\partial v}{\partial y}(x,y) &= e^x \cos(y).
\end{aligned}
$$

It follows that the Cauchy-Riemann equations are satisfied at all points in $\mathbf{R}^2$, and thus $\exp(z)$ is holomorphic on $\mathbf{C}$.

3 Power series

An important class of holomorphic functions are the power series. We will show that every complex power series defines a holomorphic function in its disk of convergence.

We start by recalling some results on power series. First we note that the definition of convergence of a power series given in [3] applies to series with complex terms.

Definition 3.1. An infinite series is an expression $\sum_{k=0}^{\infty} a_k$, where $a_k \in \mathbf{C}$. The sequence of partial sums is defined by $s_n = \sum_{k=0}^{n} a_k$. The series is convergent with sum a, if $\lim_{n\to\infty} s_n = a$. This is written as $\sum_{k=0}^{\infty} a_k = a$.

The series is said to be absolutely convergent, if the series $\sum_{k=0}^{\infty} |a_k|$ is convergent.

We recall from [3] that an absolutely convergent series is convergent. This follows in the complex case from the fact that $\mathbf{C}$ is a complete metric space.

We also recall the Weierstrass uniform convergence criterion for sequences of functions. We formulate it for complex functions defined on a subset of the complex plane.

Proposition 3.2. *Let $G \subseteq \mathbf{C}$ be a subset, and let $f_k\colon G \to \mathbf{C}$, $k \geq 0$, be a sequence of functions. Assume there exists constants $M_k \geq 0$ and $k_0 \in \mathbf{N}$, such that*

$$|f_k(z)| \leq M_k \quad \textit{for all } k \geq k_0 \textit{ and all } z \in G. \tag{3.1}$$

Assume that $\sum_{k=k_0}^{\infty} M_k$ is convergent. Then the series $\sum_{k=0}^{\infty} f_k(z)$ is absolutely and uniformly convergent on G.

Proof. This result is just a reformulation of [3, Theorem 9.29]. More precisely, we have

$$|\sum_{k=k_1}^{k_2} f_k(z)| \leq \sum_{k=k_1}^{k_2} |f_k(z)| \leq \sum_{k=k_1}^{k_2} M_k$$

for $k_2 > k_1 \geq k_0$. Thus we can apply [3, Theorem 9.29]. □

We need some further concepts from the theory of real sequences to formulate the results on power series. A real sequence $\{c_n\}$ is non-decreasing, if $c_n \leq c_{n+1}$ for all $n \in \mathbf{N}$. We have from [3, Section 2.3] the result that such a sequence is convergent, if and only if it is bounded above. In that case is converges to $\sup\{c_n\}$. We extend the concept of convergence a little by saying that it converges to $+\infty$, if it is not bounded above. This is written as $\lim_{n\to\infty} c_n = +\infty$. Analogously we have for a non-indreasing sequence, $c_n \geq c_{n+1}$, that it is convergent to a finite number if and only if it is bounded below. If it is not bounded below, we write $\lim_{n\to\infty} c_n = -\infty$. With this extension all monotone sequences of real numbers are convergent. We also extend the usage of infinum and supremum to the case of unbounded sets, allowing the values $-\infty$ and $+\infty$.

We now introduce the definitions of limes superior and limes inferior of any real sequence.

Definition 3.3. Let $\{a_n\}$ be an arbitrary real sequence. Let $r_n = \sup\{a_k \mid k \geq n\}$ and define

$$\limsup_{n\to\infty} a_n = \lim_{n\to\infty} r_n. \tag{3.2}$$

Let $t_n = \inf\{a_k \mid k \geq n\}$ and define

$$\liminf_{n\to\infty} a_n = \lim_{n\to\infty} t_n. \tag{3.3}$$

These definitions make sense, since the sequence $\{r_n\}$ is non-decreasing, and the sequence $\{t_n\}$ is non-increasing. We should note that with the extended usage the values of $\limsup_{n\to\infty} a_n$ and $\liminf_{n\to\infty} a_n$ may be $-\infty$ or $+\infty$.

We note that the definitions imply

$$\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n \tag{3.4}$$

We note the following result, which is an immediate consequence of the definition.

Proposition 3.4. *Let $\{a_n\}$ be an arbitrary real sequence.*

(a) *Let $a = \limsup_{n\to\infty} a_n$. If $r > a$, then there exists n_0, such that $a_n < r$ for all $n \geq n_0$. If $r < a$, then there exist infinitely many indices n, such that $a_n > r$.*

(b) *Let $b = \liminf_{n\to\infty} a_n$. If $r < b$, then there exists n_0, such that $r < a_n$ for all $n \geq n_0$. If $r > b$, then there exist infinitely many indices n, such that $a_n < r$.*

Theorem 3.5. *Let $\{a_n\}$ be a real sequence and $a \in \mathbf{R}$. Then $\lim_{n\to\infty} a_n = a$, if and only if $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$.*

Proof. We first assume that $\lim_{n\to\infty} a_n = a$. Let $\varepsilon > 0$ be arbitrary, and determine n_0, such that

$$a - \varepsilon < a_n < a + \varepsilon \quad \text{for all } n \geq n_0.$$

Writing again $r_n = \sup\{a_k \mid k \geq n\}$, we get that

$$a - \varepsilon \leq r_n \leq a + \varepsilon \quad \text{for all } n \geq n_0.$$

Since ε is arbitrary, we conclude that $a = \lim_{n\to\infty} r_n$, such that

$$a = \limsup_{n\to\infty} a_n.$$

Similarly, we get with $t_n = \inf\{a_k \mid k \geq n\}$ that

$$a - \varepsilon \leq t_n \leq a + \varepsilon \quad \text{for all } n \geq n_0,$$

such that $a = \lim_{n\to\infty} t_n$ and then

$$a = \liminf_{n\to\infty} a_n.$$

Now assume that $\limsup_{n\to\infty} a_n = \liminf_{n\to\infty} a_n = a$. Let $\varepsilon > 0$ be arbitrary. Since $a = \lim_{n\to\infty} t_n$, we can determine n_1, such that

$$a - \varepsilon < t_n < a + \varepsilon \quad \text{for all } n \geq n_1.$$

It follows from the definition of t_n that we have

$$a - \varepsilon < a_n \quad \text{for all } n \geq n_1.$$

Analogously, we can determine n_2 such that

$$a - \varepsilon < r_n < a + \varepsilon \quad \text{for all } n \geq n_2.$$

It follows from the definition of r_n that we have

$$a_n < a + \varepsilon \quad \text{for all } n \geq n_2.$$

Taking $n_0 = \max\{n_1, n_2\}$ we conclude that

$$a - \varepsilon < a_n < a + \varepsilon \quad \text{for all } n \geq n_0.$$

Thus since $\varepsilon > 0$ is arbitrary, we have shown $\lim_{n\to\infty} a_n = a$. □

We need one more convention. We define $1/+\infty = 0$ and $1/0 = +\infty$. We can now state a main result on power series.

Theorem 3.6. *Let $a_n \in \mathbf{C}$ and $z_0 \in \mathbf{C}$ and consider the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$. Define the extended number r, $0 \le r \le +\infty$, by*

$$r = \frac{1}{\limsup_{n\to\infty} |a_n|^{1/n}}. \tag{3.5}$$

Then the following results hold:

(a) *For $|z - z_0| < r$ the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges absolutely.*

(b) *For $|z - z_0| > r$ the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ is divergent.*

(c) *For $0 < r_1 < r$ the power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ converges uniformly on $\{z \mid |z-z_0| \le r_1\}$.*

The number r is the only number having the two properties (a) *and* (b).

Proof. It suffices to prove the result in the case $z_0 = 0$, so we impose this condition in the proof. Let $z \in \mathbf{C}$ satisfy $|z| < r$. Take t such that $|z| < t < r$. Then $1/r < 1/t$ and we can then determine n_0 such that $|a_n|^{1/n} < 1/t$ for all $n \ge n_0$. Then we get

$$|a_n z^n| < \Big(\frac{|z|}{t}\Big)^n.$$

Now $|z|/t < 1$, such that we can use Proposition 3.2 with $f_n(z) = a_n z^n$, $M_n = (|z|/t)^n$, and $G = \{z\}$.

Let now r_1 be given, with $0 < r_1 < r$. Choose t such that $r_1 < t < r$. Repeating the estimates above we get existence of an integer n_0, such that $|a_n| < (1/t)^n$ for $n \ge n_0$, and then we have

$$|a_n z^n| < \Big(\frac{r_1}{t}\Big)^n \quad \text{for all } n \ge n_0 \text{ and all } |z| \le r_1.$$

The ansolute and uniform convergence of the power series on $G = \{z \mid |z| \le r_1\}$ then follows from Proposition 3.2. Thus we have proved parts (a) and (c). To prove (b), assume $|z| > r$ and choose t with $|z| > t > r$. It follows from Proposition 3.4 that there are infinitely many indices n with $1/t < |a_n|^{1/n}$. Thus we have

$$|a_n z^n| > \Big(\frac{|z|}{t}\Big)^n$$

for infinitely many n. Now $|z|/t > 1$, so the sequence $|a_n z^n|$ is not bounded above, and therefore the power series diverges. This proves part (b).

The uniqueness statement is trivial. □

The number r given by (3.5) is called the *radius of convergence* of the power series. The ball $B(z_0, r)$ is called the ball of convergence.

We now prove that the function given by a power series with a positive radius of convergence is holomorphic in its ball of convergence. We start with a Lemma.

Lemma 3.7. *Let $\{a_n\}$ and $\{b_n\}$ be sequences of real numbers. Assume $a_n \ge 0$ for all n and furthermore that $\{b_n\}$ is convergent, $b = \lim_{n\to\infty} b_n$, where $0 < b < \infty$. Then the following result holds.*

$$\limsup_{n\to\infty}(a_n b_n) = b(\limsup_{n\to\infty} a_n) \tag{3.6}$$

Proof. Consider first the case $\limsup_{n\to\infty} a_n = \infty$. We can find N_1 such that $b_n > b/2$ for all $n \geq N_1$. Fix $K > 0$. Find N_2 such that given $n \geq N_2$ we can find $m > n$ with $a_m > (2K/b)$. But this implies that for any $n \geq \max\{N_1, N_2\}$ there exists $m > n$ such that $a_m b_m > (2K/b)(b/2) = K$. Thus the result is proved in this first case.

Assume now $a = \limsup_{n\to\infty} a_n$. Let $\varepsilon > 0$ be given. Find an $\varepsilon_1 > 0$ such that $\varepsilon_1(a+b)+\varepsilon_1^2 < \varepsilon$. Next find N such that $b - \varepsilon_1 < b_n < b + \varepsilon_1$ and $a_n < a + \varepsilon_1$ for all $n \geq N$. It follows that we have

$$a_n b_n < (a + \varepsilon_1)(b + \varepsilon_1) = ab + (\varepsilon_1(a + b) + \varepsilon_1^2) < ab + \varepsilon,$$

which shows that

$$\limsup_{n\to\infty}(a_n b_n) \leq ab + \varepsilon$$

for any $\varepsilon > 0$. For the other inequality we now find $\varepsilon_2 > 0$ such that $0 < \varepsilon_2(a + b) - \varepsilon_2^2 < \varepsilon$. Then we determine an integer N with following two properties: (i) For all $n \geq N$ we have $b - \varepsilon_2 < b_n < b + \varepsilon_2$. (ii) Given $n \geq N$, there exists an $m > n$ with $a_m > a - \varepsilon_2$. For this m we have then

$$a_m b_m > (a - \varepsilon_2)(b - \varepsilon_2) = ab - (\varepsilon_2(a + b) - \varepsilon_2^2) > ab - \varepsilon,$$

which implies

$$\limsup_{n\to\infty}(a_n b_n) \geq ab - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the result follows. □

Theorem 3.8. *Let $\sum_{n=0}^{\infty} a_n(z - z_0)^n$ be a power series with radius of convergence $r \in (0, \infty]$. Then the function f defined by this power series is infinitely differentiable in the complex sense in $B(z_0, r)$. The function $f^{(k)}(z)$ is given by the differentiated power series for each integer k. Furthermore, we have*

$$a_k = \frac{1}{k!} f^{(k)}(z_0), \quad k = 0, 1, 2 \ldots.$$

Proof. We can without loss of generality assume $z_0 = 0$. Furthermore, it suffices to show that f can be differentiated once in the complex sense, and that the derivative is given by the power series

$$f'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

with the same radius of convergence r. Let

$$\rho = \frac{1}{\limsup_{n\to\infty} |n a_n|^{1/(n-1)}}.$$

We want to show that $r = \rho$. Now since $\lim_{n\to\infty} n^{1/(n-1)} = 1$, we can use Lemma 3.7 to conclude that $\rho = 1/\limsup_{n\to\infty} |a_n|^{1/(n-1)}$. This number ρ is the radius of convergence of the power series

$$\sum_{n=1}^{\infty} a_n z^{n-1} = \sum_{n=0}^{\infty} a_{n+1} z^n.$$

Now note that we have the identity

$$\sum_{k=0}^{n+1} a_k z^k = a_0 + z \sum_{k=0}^{n} a_{k+1} z^k.$$

Let us first assume that $|z| < \rho$. Then for any integer n we have

$$\sum_{k=0}^{n+1} |a_k z^k| \le |a_0| + |z| \sum_{k=0}^{n} |a_{k+1} z^k| \le |a_0| + |z| \sum_{k=0}^{\infty} |a_{k+1} z^k| < \infty,$$

which implies that the series $\sum_{k=0}^{\infty} a_k z^k$ is absolutely convergent. We conclude (see [3]) that $\rho \le r$.

Assume now $0 < |z| < r$. Then we have for any integer n

$$\sum_{k=0}^{n} |a_{k+1} z^k| \le \frac{|a_0|}{|z|} + \frac{1}{|z|} \sum_{k=0}^{n+1} |a_k z^k| \le \frac{|a_0|}{|z|} + \frac{1}{|z|} \sum_{k=0}^{\infty} |a_k z^k| < \infty,$$

which implies $r \le \rho$. Thus we have shown that $r = \rho$.

Let us now prove differentiability in the complex sense. For a z satisfying $|z| < r$ we now define

$$g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}, \quad s_n(z) = \sum_{k=0}^{n} a_k z^k, \quad R_n(z) = \sum_{k=n+1}^{\infty} a_k z^k,$$

such that $f(z) = s_n(z) + R_n(z)$. Fix z_1 with $|z_1| < r$. We want to prove that f is differentiable at z_1 with derivative $g(z_1)$. Fix r_1 such that $|z_1| < r_1 < r$. Next determine a $\delta > 0$ such that $B(z_1, \delta) \subset B(0, r_1)$. Now let $z \in B(z_1, \delta)$, $z \neq z_1$. We have

$$\begin{aligned} \frac{f(z) - f(z_1)}{z - z_1} - g(z_1) &= \frac{s_n(z) - s_n(z_1)}{z - z_1} - s_n'(z_1) \\ &\quad + s_n'(z_1) - g(z_1) + \frac{R_n(z) - R_n(z_1)}{z - z_1}. \end{aligned}$$

The last term is rewritten as

$$\begin{aligned} \frac{R_n(z) - R_n(z_1)}{z - z_1} &= \frac{1}{z - z_1} \sum_{k=n+1}^{\infty} a_k (z^k - z_1^k) \\ &= \sum_{k=n+1}^{\infty} a_k \Big(\frac{z^k - z_1^k}{z - z_1} \Big). \end{aligned}$$

We now estimate as follows

$$\frac{|z^k - z_1^k|}{|z - z_1|} = |z^{k-1} + z^{k-2} z_1 + \ldots + z z_1^{k-2} + z_1^{k-1}| \le k r_1^{k-1}. \tag{3.7}$$

Thus

$$\left| \frac{R_n(z) - R_n(z_1)}{z - z_1} \right| \le \sum_{k=n+1}^{\infty} |a_k| k r_1^{k-1}.$$

The series $\sum_{k=0}^{\infty} |a_k| k r_1^{k-1}$ is convergent, since $r_1 < r$. Given $\varepsilon > 0$, we can determine N_1 such that for $n \ge N_1$ we have

$$\left| \frac{R_n(z) - R_n(z_1)}{z - z_1} \right| < \frac{\varepsilon}{3}.$$

Since $\lim_{n\to\infty} s_n'(z_1) = g(z_1)$, we can determine N_2 such that $|s_n'(z_1) - g(z_1)| < \varepsilon/3$ for all $n \ge N_2$. Now choose a fixed n given by $n = \max\{N_1, N_2\}$. The polynomial $s_n(z)$ is clearly differentiable, so we can find $\mu > 0$ such that

$$\left| \frac{s_n(z) - s_n(z_1)}{z - z_1} - s_n'(z_1) \right| < \frac{\varepsilon}{3}$$

for all z satisfying $0 < |z - z_1| < \mu$. If we combine the estimates, we have shown that

$$\left| \frac{f(z) - f(z_1)}{z - z_1} - g(z_1) \right| < \varepsilon$$

for all z satisfying $0 < |z - z_1| < \min\{\delta, \mu\}$. Thus we have shown differentiability in the complex sense at an arbitrary $z \in B(0, r_1)$, and since this holds for any $r_1 < r$, we have differentiability in the ball $B(0, r)$. □

We introduce the following definition:

Definition 3.9. Let $G \subseteq \mathbf{C}$ be an open subset. A function $f : G \to \mathbf{C}$ is said to be analytic in G, if for every $z_0 \in G$ there exist an $r > 0$ and a power series $\sum_{n=0}^{\infty} a_n (z - z_0)^n$ whose sum equals $f(z)$ in $B(z_0, r) \subseteq G$.

Combining Definition 3.9 and Theorem 3.8 we can state the following result.

Proposition 3.10. *Let $G \subseteq \mathbf{C}$ be an open subset. If a function f from G to $\mathbf{C}$ is analytic, then it is holomorphic.*

Stated briefly, analytic functions are holomorphic functions. One of the main results in complex analysis is the converse, namely that every holomorphic function is analytic. The first rigorous treatment of complex analysis was given by K. Weierstrass (1815–1897). He based his approach on the concept of an analytic function. Later presentations, including the one given here, base their study on the concept of a holomorphic function.

3.1 Exercises

1. Prove that every complex polynomial $p(z) = a_0 + a_1 z + \ldots + a_n z^n$ is holomorphic on $\mathbf{C}$.
2. Prove that $\sin(z)$ and $\cos(z)$ are holomorphic on $\mathbf{C}$.
3. Verify that the proofs for the rules of differentiation in the real case, as for example given in [3, Chapter 4], are valid in the complex case.
4. At which points are the following functions differentiable in the complex sense?

$$\text{(a) } f(z) = y, \quad \text{(b) } f(z) = \overline{z}, \quad \text{(c) } f(z) = \overline{z}^2.$$

5. Prove that the function $f(z) = \sqrt{|xy|}$ is not differentiable in the complex sense at the origin, even though it satisfies the Cauchy-Riemann equations at that point.
6. Give the details in the argument leading to Theorem 2.6.
7. Assume that $G \subseteq \mathbf{C}$ is open and connected, and that $f \in \mathcal{H}(G)$. Assume $f'(z) = 0$ for all $z \in G$. Prove that f is constant on G.
8. Verify the equality in (3.7).
9. Assume $f \in \mathcal{H}(G)$. Define $g(z) = \overline{f(\overline{z})}$ for $z \in \overline{G}$ (the subset consisting of all complex conjugates of points in G). Show that $g \in \mathcal{H}(\overline{G})$.
10. Prove Proposition 3.4.

4 Contour Integrals

A fundamental tool in the study of complex functions is the contour integral (or complex line integral). We proceed to give the various definitions. The reader should note that terminology concerning curves and paths is not consistent in the mathematical literature.

Definition 4.1. A path in the complex plane is a continuous function $\gamma\colon [a,b] \to \mathbf{C}$. The path is said to be closed, if $\gamma(a) = \gamma(b)$. The path is said to be simple, if the restriction of γ to $[a,b)$ is injective. The image of γ is denoted by γ^*, i.e. $\gamma^* = \gamma([a,b])$.

Given the image of a path in the complex plane, there can be many other paths having the same image. We introduce the following equivalence relation.

Definition 4.2. Let $\gamma\colon [a,b] \to \mathbf{C}$ and $\tau\colon [c,d] \to \mathbf{C}$ be two paths in the complex plane with $\gamma^* = \tau^*$. The paths γ and τ are said to be equivalent, if there exists a continuous strictly increasing function φ from $[a,b]$ onto $[c,d]$ such that $\tau \circ \varphi = \gamma$.

We want to define a contour integral along a path. For this purpose we need a restricted class of paths. The continuous function $\gamma\colon [a,b] \to \mathbf{C}$ is said to be piecewise smooth, if there exists a finite partition $a = t_0 < t_1 < \cdots < t_n = b$ such that the restriction of γ to $[t_{j-1}, t_j]$ is continuously differentiable for $j = 1, \ldots, n$. Note that the derivatives $\gamma'(t_j-)$ can be different from $\gamma'(t_j+)$. The function $|\gamma'(t)|$ is not defined at the points t_j, but it is bounded and continuous on (t_{j-1}, t_j), and has limits at the end points. Since the values at a finite number of points are irrelevant in the definition of the Riemann integral, we conclude that $|\gamma'(t)|$ is Riemann integrable over $[a,b]$.

Definition 4.3. A path $\gamma\colon [a,b] \to \mathbf{C}$ is called a circuit, if the function γ is piecewise smooth.

Definition 4.4. Let $\gamma\colon [a,b] \to \mathbf{C}$ and $\tau\colon [c,d] \to \mathbf{C}$ be two circuits with $\gamma^* = \tau^*$. They are said to be equivalent, if there exists a continuously differentiable and strictly increasing function φ from $[a,b]$ onto $[c,d]$ such that $\tau \circ \varphi = \gamma$.

When two circuits γ and τ are equivalent, we say that τ is a reparametrization of γ.

Definition 4.5. Let γ be a circuit in the complex plane. Then the length of this circuit is given by $L(\gamma) = \int_a^b |\gamma'(t)| dt$.

The length is independent of parametrization, see Remark 4.12.

Let us note that our circuits are oriented. The parametrization determines the orientation. A simple closed path can be either positively or negatively oriented, i.e. oriented in the counterclockwise or the clockwise direction.

Example 4.6. The unit circle $S^1 = \{z \in \mathbf{C} \mid |z| = 1\}$ is the image of many different paths and circuits. Consider the following five circuits:

$$
\begin{aligned}
\gamma_1(t) &= e^{it}, && t \in [0, 2\pi], \\
\gamma_2(t) &= e^{-it}, && t \in [0, 2\pi], \\
\gamma_3(t) &= e^{2it}, && t \in [0, \pi], \\
\gamma_4(t) &= e^{3it}, && t \in [0, 2\pi], \\
\gamma_5(t) &= e^{i|t|}, && t \in [-2\pi, 2\pi].
\end{aligned}
$$

Only the circuits γ_1 and γ_3 are equivalent, with $\varphi(t) = t/2$.

The circuits γ_j, $j = 1, 2, 3$, are all simple closed paths. The circuit γ_4 is closed, but not simple, since $\gamma_4(0) = \gamma_4(2\pi/3) = \gamma_4(4\pi/3) = \gamma_4(2\pi)$. The circuit γ_5 is closed, but not simple, since $\gamma_5(-2\pi) = \gamma_5(0) = \gamma_5(2\pi)$.

As t varies from 0 to 2π, the point $\gamma_1(t)$ traverses the unit circle once in the positive direction, and $\gamma_2(t)$ once in the negative direction, whereas $\gamma_4(t)$ traverses the unit circle three times in the positive direction.

A circle in the complex plane is often described as the boundary of a ball, with the notation $\partial B(a, r)$. Viewing this boundary as a circuit, our convention is to assume that this circuit is given by

$$\gamma(t) = a + re^{it}, \quad t \in [0, 2\pi].$$

The line segment from z to w in the complex plane is denoted by $L(z, w)$. Viewed as a circuit our convention is that this circuit is given by

$$\gamma(t) = z + t(w - z), \quad t \in [0, 1].$$

Example 4.7. Let Δ be a triangle with vertices $a, b, c \in \mathbf{C}$. The boundary $\partial\Delta$ is viewed as a circuit. One possible circuit is given as follows:

$$\gamma(t) = \begin{cases} a + t(b - a), & t \in [0, 1], \\ b + (t - 1)(c - b), & t \in [1, 2], \\ c + (t - 2)(a - c), & t \in [2, 3]. \end{cases}$$

The orientation depends on the relative location of the three vertices. Note that we are not excluding the degenerate cases, where vertices coincide or lie on a straight line.

Example 4.8. A polygonal circuit is a circuit composed of a finite number of line segments $L(z_1, z_2)$, $L(z_2, z_3), \ldots, L(z_{n-1}, z_n)$. A parametrization can be given as follows:

$$\gamma(t) = \begin{cases} z_1 + t(z_2 - z_1), & t \in [0, 1], \\ z_2 + (t - 1)(z_3 - z_2), & t \in [1, 2], \\ \quad\vdots & \quad\vdots \\ z_{n-1} + (t - n + 2)(z_n - z_{n-1}), & t \in [n - 2, n - 1]. \end{cases}$$

Given two polygonal circuits γ_1 and γ_2, such that the end point of γ_1 equals the starting point of γ_2, then we denote by $\gamma_1 \cup \gamma_2$ the concatenation of the two circuits. It is again a polygonal circuit. Finally we define the contour integrals.

Definition 4.9. Let $\gamma\colon [a, b] \to \mathbf{C}$ be a circuit. Let $f\colon \gamma^* \to \mathbf{C}$ be a continuous function. The contour integral of f along γ is defined by

$$\int_\gamma f(z)dz = \int_a^b f(\gamma(t))\gamma'(t)dt.$$

Often we simplify the notation and write $\int_\gamma f$ instead of $\int_\gamma f(z)dz$.

We recall from [3] that we have defined the Riemann integrability of a complex function as the joint Riemann integrability of the real and imaginary parts. We recall some results from [3]. The space of Riemann integrable complex functions defined on $[a, b]$ is denoted by $\mathcal{R}([a, b], \mathbf{C})$.

Proposition 4.10. *$\mathcal{R}([a,b],\mathbf{C})$ is a complex vector space. For $f_1, f_2 \in \mathcal{R}([a,b],\mathbf{C})$ and $c_1, c_2 \in \mathbf{C}$ the following results hold:*

$$\int_a^b (c_1 f_1(t) + c_2 f_2(t))dt = c_1 \int_a^b f_1(t)dt + c_2 \int_a^b f_2(t)dt, \tag{4.1}$$

$$\left| \int_a^b f_1(t)dt \right| \leq \int_a^b |f_1(t)|\, dt. \tag{4.2}$$

We need to show that the definition of the contour integral is independent of the choice of parametrization.

Theorem 4.11. *The value of a contour integral is unchanged under reparametrization of the circuit.*

Proof. Let $\gamma^* = \tau^*$ be a parametrization and a reparametrization of the circuit. By Definition 4.4 we have $\tau \circ \varphi = \gamma$. To simplify the proof we assume that all three functions are continuously differentiable on their definition intervals. Using change of variables for Riemann integrals (see [3, Section 7.2]) and the chain rule we find

$$\begin{aligned}\int_c^d f(\tau(s))\tau'(s)ds &= \int_a^b f(\tau(\varphi(t)))\tau'(\varphi(t))\varphi'(t)dt \\ &= \int_a^b f(\gamma(t))\gamma'(t)dt.\end{aligned}$$

This computation finishes the proof in the simplified case. In the general case, where γ and τ are piecewise smooth, the integral is split into a sum over the intervals of smoothness, and the above computation is performed on each interval. □

Remark 4.12. A similar computation shows that the length of a circuit is unchanged under reparametrization.

The following estimate is used several times in the sequel.

Proposition 4.13. *Let $\gamma\colon [a,b] \to \mathbf{C}$ be a circuit. Let $f\colon \gamma^* \to \mathbf{C}$ be a continuous function. Then we have*

$$\left| \int_\gamma f(z)dz \right| \leq \max_{\gamma^*} |f| \cdot L(\gamma).$$

As a consequence, if $f_n \to f$ uniformly on γ^*, then $\int_\gamma f_n \to \int_\gamma f$.

Proof. This estimate follows from the computation

$$\begin{aligned}\left| \int_\gamma f(z)dz \right| &= \left| \int_a^b f(\gamma(t))\gamma'(t)dt \right| \\ &\leq \int_a^b |f(\gamma(t))| \cdot |\gamma'(t)| dt \leq \max_{\gamma^*} |f| \cdot L(\gamma),\end{aligned}$$

where we used (4.2) and Definition 4.5. □

We now look at primitives of complex functions, and their use in evaluation of contour integrals.

Definition 4.14. An open and connected subset $G \subseteq \mathbf{C}$ is called a domain.

Definition 4.15. Let $f\colon G \to \mathbf{C}$ be defined on a domain G. A function $F\colon G \to \mathbf{C}$ is called a primitive of f, if $F \in \mathcal{H}(G)$ and $F' = f$.

If F is a primitive of f, then $F + c$ is also a primitive of f for all $c \in \mathbf{C}$. Conversely, assume that F_1 and F_2 both are primitives of f. Then $(F_1 - F_2)' = f - f = 0$ on G, and since G is assumed to be connected, it follows from Theorem 2.5 that $F_1 - F_2$ is constant on G. Thus the primitive is determined uniquely up to an additive constant.

Theorem 4.16. *Assume that G is a domain and that $f: G \to \mathbf{C}$ is a continuous function. Assume that F is a primitive of f in G. Then*

$$\int_\gamma f(z)dz = F(z_2) - F(z_1)$$

for any circuit in G from z_1 to z_2.

Proof. The result follows from the computation

$$\begin{aligned}
\int_\gamma f(z)dz &= \int_a^b \mathrm{Re}[f(\gamma(t))\gamma'(t)]dt + i\int_a^b \mathrm{Im}[f(\gamma(t))\gamma'(t)]dt \\
&= \int_a^b \mathrm{Re}[\frac{d}{dt}F(\gamma(t))]dt + i\int_a^b \mathrm{Im}[\frac{d}{dt}F(\gamma(t))]dt \\
&= \int_a^b \frac{d}{dt}\mathrm{Re}[F(\gamma(t))]dt + i\int_a^b \frac{d}{dt}\mathrm{Im}[F(\gamma(t))]dt \\
&= \mathrm{Re}[F(\gamma(b))] - \mathrm{Re}[F(\gamma(a))] + i\,\mathrm{Im}[F(\gamma(b))] - i\,\mathrm{Im}[F(\gamma(a))] \\
&= F(\gamma(b)) - F(\gamma(a)),
\end{aligned}$$

where we used results on the Riemann integral [3]. □

Theorem 4.17. *Let $f: G \to \mathbf{C}$ be a continuous function on a domain $G \subseteq \mathbf{C}$. Assume that $\int_\gamma f = 0$ for any closed polygonal circuit in G. Then f has a primitive in G.*

Proof. Choose a point $z_0 \in G$ and define $F(z) = \int_{\gamma_z} f(\zeta)d\zeta$, where γ_z is a polygonal circuit from z_0 to z in G. Note that such a circuit exists due to results on connectedness, see [2]. Our assumption implies that the value $F(z)$ is independent of the choice of such a circuit.

Given a $z \in G$ there exists $r > 0$ such that the ball $B(z, r) \subseteq G$. Let $h \in \mathbf{C}$ satisfy $0 < |h| < r$, and let ℓ be the line segment from z to $z + h$. Then

$$F(z+h) - F(z) = \int_{\gamma_z \cup \ell} f - \int_{\gamma_z} f = \int_\ell f = \int_0^1 f(z + th)h\,dt$$

and thus

$$\frac{1}{h}(F(z+h) - F(z)) - f(z) = \int_0^1 (f(z+th) - f(z))dt.$$

Since f is continuous, we can to a given $\varepsilon > 0$ determine a $\delta > 0$ such that $|f(w) - f(z)| < \varepsilon$ for all $w \in B(z, \delta)$, and therefore

$$\left|\frac{1}{h}(F(z+h) - F(z)) - f(z)\right| \le \int_0^1 \varepsilon dt = \varepsilon, \quad 0 < |h| < \delta.$$

Thus F is differentiable at z with $F'(z) = f(z)$. Since $z \in G$ was arbitrary, the result is proved. □

Let us show in an example how to compute a contour integral.

Example 4.18. Let C_r denote the circle $|z| = r$ traversed once in the positive direction (counterclockwise).

$$C_r(t) = re^{it}, \quad t \in [0, 2\pi].$$

Then for each integer $n \in \mathbf{Z}$ we have

$$\int_{C_r} \frac{dz}{z^n} = \int_0^{2\pi} \frac{rie^{it}}{r^n e^{int}} dt = ir^{1-n} \int_0^{2\pi} e^{it(1-n)} dt = \begin{cases} 0, & n \neq 1, \\ 2\pi i, & n = 1. \end{cases}$$

The result can also be obtained in the case $n \neq 1$ by observing that the function z^{-n} has as its primitive the function $(1-n)^{-1} z^{1-n}$, in $\mathbf{C}$ for any $n \leq 0$, and in $\mathbf{C} \setminus \{0\}$ for $n \geq 2$. Since the integral is nonzero for $n = 1$, we can conclude that the function z^{-1} has no primitive in $\mathbf{C} \setminus \{0\}$.

4.1 Exercises

1. Carry out all the details in the three examples 4.6, 4.7, and 4.8.

2. Compute the following contour integrals:

$$\int_0^i \frac{dz}{(1-z)^2}, \qquad \int_i^{2i} \cos(z) dz, \quad \text{and} \quad \int_0^{i\pi} e^z dz,$$

   where in each case the circuit is the line segment from the lower limit to the upper limit. Repeat the computations using a primitive for the integrand in each of the three integrals.

3. Show that

$$\int_\gamma \frac{z}{(z^2+1)^2} dz = 0,$$

   for any closed circuit γ in $\mathbf{C} \setminus \{\pm i\}$.

4. Show that

$$\int_\gamma P(z) dz = 0,$$

   for any polynomial $P(z)$, and any closed circuit γ in $\mathbf{C}$.

5 Cauchy's theorems

In this section we first study the question of existence of a primitive to a given holomorphic function. Example 4.18 shows that a primitive need not exist. The existence of a primitive depends on both the function and the domain we consider. One can obtain existence of a primitive for any $f \in \mathcal{H}(G)$ by imposing a simple geometric condition on the domain G.

Definition 5.1. A domain $G \subseteq \mathbf{C}$ is said to be starshaped around $a \in G$, if for all $z \in G$ the line segment $L(a, z) = \{a + t(z - a) \mid t \in [0, 1]\} \subseteq G$. The domain is called starshaped, if there exists at least one such $a \in G$.

We will now prove that if a domain G is starshaped, then any holomorphic function on G has a primitive in G. The starting point is the following Lemma.

Lemma 5.2 (Goursat's lemma (1899))**.** *Let $G \subseteq \mathbf{C}$ be an open subset, and assume that $f \in \mathcal{H}(G)$. Then*

$$\int_{\partial\Delta} f(z) dz = 0$$

for any solid triangle $\Delta \subseteq G$.

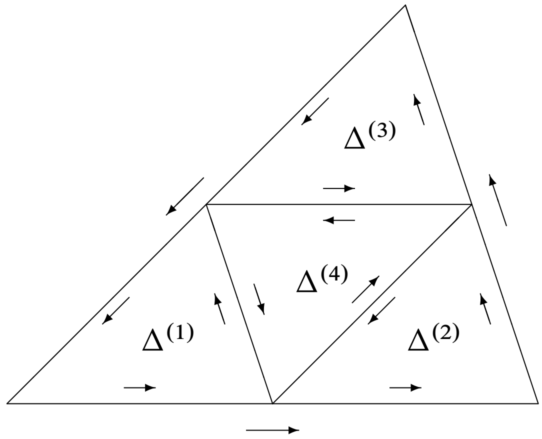

Figure 5.1: Partition of Δ.

Proof. We connect the midpoints of the sides in the triangle Δ by line segments, thus dividing the large triangle into four triangles, denoted by $\Delta^{(i)}$, see Figure 5.1.

It is easy to see that we have

$$I = \int_{\partial\Delta} f = \sum_{i=1}^{4} \int_{\partial\Delta^{(i)}} f.$$

At least one of the four contour integrals $\int_{\partial\Delta^{(i)}} f$ must have an absolute value which is greater than or equal to $|I|/4$. We select one such triangle and denote it by Δ_1. Thus we have $|I| \le 4|\int_{\partial\Delta_1} f|$. We now divide the triangle Δ_1 into four triangles by connection midpoints on the sides, as above. One of these four triangles, which we will denote by Δ_2, will satisfy $|\int_{\partial\Delta_1} f| \le 4|\int_{\partial\Delta_2} f|$. We repeat this construction, obtaining a nested decreasing sequence of triangles $\Delta \supset \Delta_1 \supset \Delta_2 \supset \cdots$, which satisfy

$$|I| \le 4^n \left| \int_{\partial\Delta_n} f \right|, \quad n = 1, 2, 3, \cdots .$$

There exists a unique z_0 such that $\cap_{n=1}^{\infty} \Delta_n = \{z_0\}$. This result is obtained by first using Cantor's theorem, which proves that the intersection is nonempty. But since the diameter of the triangles is strictly decreasing, the intersection can only contain one point.

We now use the differentiability of f at z_0 to prove that $I = 0$. We have (recall Lemma 2.3)

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + |z - z_0| E(z - z_0),$$

where $E(z - z_0) \to 0$ for $z \to z_0$. Given $\varepsilon > 0$ we can determine $\delta > 0$ such that

$$|f(z) - f(z_0) - f'(z_0)(z - z_0)| \le \varepsilon |z - z_0|, \quad \text{for all} \quad z \in B(z_0, \delta) \subseteq G.$$

Let now L_0 denote the length of the original triangle circuit $\partial\Delta$. The length of $\partial\Delta_n$ is then $2^{-n} L_0$. Thus there exists $N \in \mathbf{N}$ such that $\Delta_n \subseteq B(z_0, \delta)$ for $n \ge N$. For $z \in \partial\Delta_N$ the distance $|z - z_0|$ is at most equal to half the circumference of Δ_N, which implies $|z - z_0| \le 2^{-(N+1)} L_0$. We also note that

$$\int_{\partial\Delta_N} \big(f(z_0) + f'(z_0)(z - z_0) \big)\, dz = 0,$$

since a polynomial of degree at most one has a primitive, and the integral around a closed circuit then is zero, see Theorem 4.16. We now have the following sequence of estimates.

$$|I| \le 4^N \left| \int_{\partial\Delta_N} f(z) dz \right| = 4^N \left| \int_{\partial\Delta_N} \big(f(z) - f(z_0) - f'(z_0)(z - z_0) \big)\, dz \right|$$

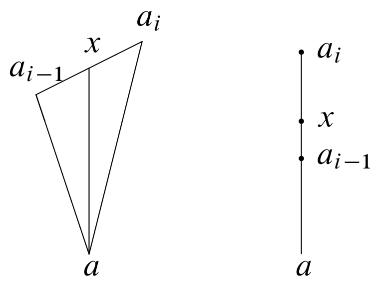

Figure 5.2: The two cases.

$$\le 4^N \max_{z\in\partial\Delta_N} \left|\left(f(z) - f(z_0) - f'(z_0)(z-z_0)\right)\right| L(\partial\Delta_N)$$
$$\le 4^N \varepsilon \max_{z\in\partial\Delta_N} |z - z_0| \, 2^{-N} L_0 \le \frac{1}{2}\varepsilon L_0^2.$$

Since $\varepsilon > 0$ is arbitrary, it follows that $I = 0$. □

Goursat's Lemma is used to prove the following important result.

Theorem 5.3 (Cauchy's integral theorem)**.** *Let G be a starshaped domain, and let $f \in \mathcal{H}(G)$. Then $\int_\gamma f(z)dz = 0$ for any closed polygonal circuit γ in G.*

Proof. Assume that G is starshaped around $a \in G$. Let γ be a closed polygonal circuit with vertices $a_0, a_1, \ldots, a_{n-1}, a_n = a_0$. Let x be an arbitrary point on one of the line segments from a_{i-1} to a_i, $i = 1, \ldots, n$. Since G is starshaped around a, the line segment $L(a, x)$ will be contained in G. Thus the solid triangle with vertices $\{a, a_{i-1}, a_i\}$, denoted by $\Delta\{a, a_{i-1}, a_i\}$, will be contained in G. The integral of f around the circuit determined by the triangle, traversed in the order from a to a_{i-1}, then from a_{i-1} to a_i, and finally from a_i to a, will be zero. If the triangle is nondegenerate, this result is an immediate consequence of Lemma 5.2. If the triangle is degenerate, which means that the three points lie on a straight line, the result is obvious. See Figure 5.2. It follows that in all cases
$$\sum_{i=1}^{n} \int_{\partial\Delta\{a,a_{i-1},a_i\}} f(z)dz = 0.$$
Each of the line segments connecting a with a_i is traversed twice, in opposite directions. If we split the integrals into integrals over line segments, then these terms cancel, and we are left with $\int_\gamma f$, which then equals zero, as claimed in the theorem. □

Combining Theorem 5.3 with Theorem 4.17, we get the following result.

Theorem 5.4. *Let $G \subseteq \mathbf{C}$ be a starshaped domain. Then any function $f \in \mathcal{H}(G)$ has a primitive in G.*

An immediate consequence is that Cauchy's integral theorem for a starshaped domain holds not just for polygonal circuits, but for any closed circuit in G.

Corollary 5.5. *Let $G \subseteq \mathbf{C}$ be a starshaped domain. Let $f \in \mathcal{H}(G)$ and let γ be a closed circuit in G. Then $\int_\gamma f(z)dz = 0$.*

Cauchy's integral theorem allows us to express the values of a holomorphic function in terms of certain contour integrals.

We start with some preliminary considerations. Let G be a domain, $z_0 \in G$, and let $f \in \mathcal{H}(G\setminus\{z_0\})$. We want to compute the contour integral of f along a closed simple circuit C in $G\setminus\{z_0\}$

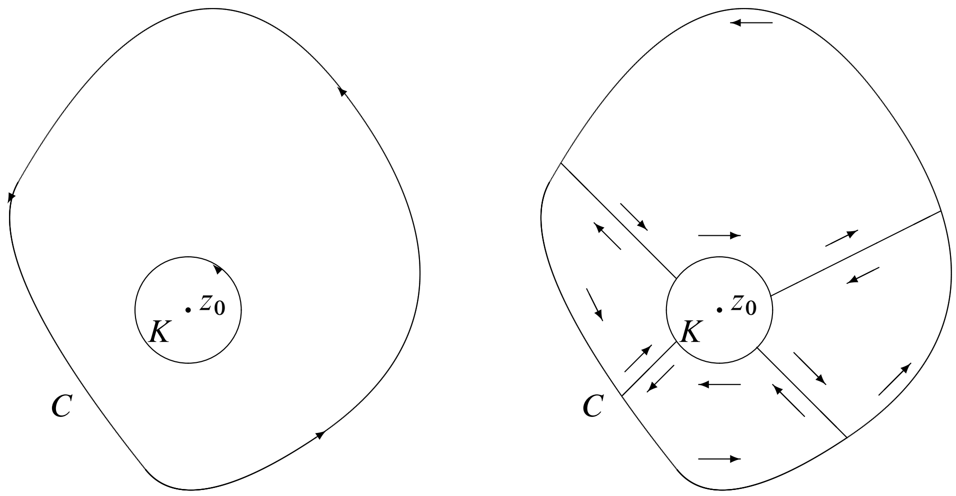

Figure 5.3: Decomposition in smaller circuits.

which encloses z_0. Let us assume that C is oriented counterclockwise. A common application of the Cauchy integral theorem is to replace the circuit C by another circuit K, positively oriented, in a contour integral. This new circuit is also assumed to enclose z_0 and to lie in G. The idea is to select certain points on C and K and connect them to obtain a number of small circuits γ_j. See Figure 5.3. Assume that we can construct a finite number of γ_j, such that each γ_j lies in a starshaped subdomain of $G \setminus \{z_0\}$. Then we get from the above generalization of Cauchy's integral theorem that we have

$$0 = \sum_i \int_{\gamma_i} f = \int_C f + \int_{-K} f,$$

or

$$\int_C f = \int_K f.$$

We have used the notation $\int_{-K} f$ to denote the integral along K in the direction opposite to the one given in the definition of K.

An important case where this construction can be performed, is described in the following example.

Example 5.6. Let G be a domain, $z_0 \in G$, and assume that $f \in \mathcal{H}(G \setminus \{z_0\})$. Assume that for some $0 < s < r$ we have $\overline{B}(z_0, s) \subseteq B(a, r)$, $\overline{B}(a, r) \subseteq G$. Then we have

$$\int_{\partial B(a,r)} f(z)dz = \int_{\partial B(z_0,s)} f(z)dz.$$

This result is obtained by the technique described, by adding four line segments parallel to the axes from $\partial B(z_0, s)$ to $\partial B(a, r)$.

We can now state one of the main results.

Theorem 5.7 (Cauchy's integral formula)**.** *Let $G \subseteq \mathbf{C}$ be an open subset, $f \in \mathcal{H}(G)$ and $\overline{B}(a, r) \subseteq G$. For all $z_0 \in B(a, r)$ we then have the formula*

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B(a,r)} \frac{f(z)}{z - z_0} dz,$$

where the circle is traversed once in the positive direction.

Proof. Let $z_0 \in B(a,r)$. Using Example 5.6 on the function $g(z) = (z-z_0)^{-1} f(z)$, which is holomorphic in $G \setminus \{z_0\}$, we find that

$$\int_{\partial B(a,r)} \frac{f(z)}{z-z_0} dz = \int_{\partial B(z_0,s)} \frac{f(z)}{z-z_0} dz$$

for $0 < s < r - |a - z_0|$. We introduce the parametrization $\gamma(t) = z_0 + se^{it}$, $t \in [0, 2\pi]$, for $\partial B(z_0, s)$. Then we find that

$$\int_{\partial B(z_0,s)} \frac{dz}{z-z_0} = \int_0^{2\pi} \frac{sie^{it}}{se^{it}} dt = 2\pi i,$$

which implies

$$I = \int_{\partial B(a,r)} \frac{f(z)}{z-z_0} dz - 2\pi i f(z_0) = \int_{\partial B(z_0,s)} \frac{f(z) - f(z_0)}{z-z_0} dz.$$

Our goal is to show that $I = 0$. Proposition 4.13 implies the estimate

$$\begin{aligned} |I| &\le \max_{z \in \partial B(z_0,s)} \{|(f(z) - f(z_0))/(z-z_0)|\} \, L(\partial B(z_0,s)) \\ &= 2\pi \max_{z \in \partial B(z_0,s)} \{|f(z) - f(z_0)|\}. \end{aligned}$$

Since f is continuous at z_0, the right hand side will tend to zero for $s \to 0$, which implies the result. □

Cauchy's integral formula implies that knowing the values of the holomorphic function f on the circle $|z-a| = r$ allows us to find the value at any point in the interior of this circle. Note that if we take $z_0 = a$ and use the parametrization $\gamma(t) = a + re^{it}$, $t \in [0, 2\pi]$, for $\partial B(a,r)$, then we get $f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a + re^{it}) dt$. In other words, the value at the center of the circle equals the mean over the values on the circumference.

Cauchy's integral formula can be used to evaluate some contour integrals.

Example 5.8. Let us show how to evaluate $\int_{\partial B(0,2)} \frac{\sin(z)}{1+z^2} dz$. We have

$$\begin{aligned} \int_{\partial B(0,2)} \frac{\sin(z)}{1+z^2} dz &= \frac{1}{2i} \int_{\partial B(0,2)} \frac{\sin(z)}{z-i} dz - \frac{1}{2i} \int_{\partial B(0,2)} \frac{\sin(z)}{z+i} dz \\ &= \pi \sin(i) - \pi \sin(-i) = 2\pi \sin(i) = \pi i (e - \frac{1}{e}). \end{aligned}$$

5.1 Exercises

1. Evaluate

$$\int_{\partial B(0,1)} \frac{dz}{(z-a)(z-b)}$$

in the following cases

(a) $|a| < 1$ and $|b| < 1$.

(b) $|a| < 1$ and $|b| > 1$.

(c) $|a| > 1$ and $|b| > 1$.

2. Evaluate
$$\int_{\partial B(0,2)} \frac{e^z}{z-1} dz \quad \text{and} \quad \int_{\partial B(0,2)} \frac{e^z}{\pi i - 2z} dz.$$
3. Give a detailed proof of the result stated in Example 5.6.
4. Cantor's theorem in $\mathbf{R}^2$ states the following. Let A_k, $k \in \mathbf{N}$ be a sequence of non-empty subsets of $\mathbf{R}^2$ with the following two properties
   (a) $A_{k+1} \subseteq A_k, k = 1, 2, \ldots$.
   (b) Each A_k is closed and non-empty. A_1 is bounded.

   Then $\cap_{k=1}^{\infty} A_k \neq \emptyset$.

   Prove this theorem.

6 Applications of Cauchy's integral formula

Let $G \subseteq \mathbf{C}$ be an open subset, and fix $a \in G$. In the case $G = \mathbf{C}$ we let $\rho = \infty$ and $B(a, \rho) = \mathbf{C}$. In the case $G \neq \mathbf{C}$ we let $\rho = \min\{|z - a| \,|\, z \in \mathbf{C} \setminus G\}$. Then in all cases the ball $B(a, \rho)$ is the largest ball centered at a and contained in G.

We will now use Cauchy's integral formula to prove that a function $f \in \mathcal{H}(G)$ is analytic. More precisely, we will prove that for any $a \in G$ the Taylor expansion
$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n \tag{6.1}$$
is convergent in the largest ball $B(a, \rho)$ contained in G, and the sum equals $f(z)$ for each $z \in B(a, \rho)$

Theorem 6.1. *Let $G \subseteq \mathbf{C}$ be an open subset, and let $f \in \mathcal{H}(G)$. Then f is infinitely often differentiable in the complex sense, and the Taylor expansion* (6.1) *is convergent with sum f in the largest open ball $B(a, \rho)$ contained in G.*

Proof. The function $(z-a)^{-(n+1)} f(z)$ is holomorphic in $G \setminus \{a\}$. Example 5.6 implies that the numbers
$$a_n = \frac{1}{2\pi i} \int_{\partial B(a,r)} \frac{f(z)}{(z-a)^{n+1}} dz, \qquad n = 0, 1, 2, \cdots$$
are independent of r for $0 < r < \rho$. For a given fixed $z_0 \in B(a, \rho)$ we choose r satisfying $|z_0 - a| < r < \rho$. Cauchy's integral formula implies
$$f(z_0) = \frac{1}{2\pi i} \int_{\partial B(a,r)} \frac{f(z)}{z - z_0} dz.$$
The idea in the proof is to rewrite the integrand as a convergent series and integrate term by term. Let $z \in \partial B(a, r)$. Note that
$$\left|\frac{z_0 - a}{z - a}\right| = \frac{|z_0 - a|}{r} < 1. \tag{6.2}$$
We have
$$\frac{1}{z - z_0} = \frac{1}{z - a + a - z_0} = \frac{1}{z - a} \frac{1}{1 - \frac{z_0 - a}{z - a}} = \frac{1}{z - a} \sum_{n=0}^{\infty} \left(\frac{z_0 - a}{z - a}\right)^n,$$

which implies (with the obvious definition of $g_n(z)$)

$$\frac{f(z)}{z-z_0} = \sum_{n=0}^{\infty} \frac{f(z)(z_0-a)^n}{(z-a)^{n+1}} = \sum_{n=0}^{\infty} g_n(z).$$

Since $|f(z)|$ is a continuous function on the compact set $\partial B(a,r)$, it has a maximal value $M < \infty$ on this set. Thus we have for all $z \in \partial B(a,r)$

$$|g_n(z)| \le \frac{M}{r}\left(\frac{|z_0-a|}{r}\right)^n, \qquad \sum_{n=0}^{\infty}\left(\frac{|z_0-a|}{r}\right)^n < \infty,$$

see (6.2). The Weierstrass M-test Proposition 3.2 implies that the series $\sum_{n=0}^{\infty} g_n(z)$ converges uniformly on $\partial B(a,r)$. Thus we can integrate term by term.

$$f(z_0) = \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(z)}{z-z_0}dz = \sum_{n=0}^{\infty} \frac{1}{2\pi i}\int_{\partial B(a,r)} g_n(z)dz = \sum_{n=0}^{\infty} a_n(z_0-a)^n.$$

We have shown that the power series $\sum_{n=0}^{\infty} a_n(z_0-a)^n$ is convergent with sum $f(z_0)$ for all $z_0 \in B(a,\rho)$. It follows from Theorem 3.8 that f is infinitely differentiable in the complex sense, and furthermore that the coefficients a_n above are given by $(n!)^{-1} f^{(n)}(a)$. □

One consequence of this theorem is important enough to state separately.

Corollary 6.2. *Assume $f \in \mathcal{H}(G)$. Then $f' \in \mathcal{H}(G)$.*

Using this result we can state the following important theorem.

Theorem 6.3 (Morera)**.** *Let $G \subseteq \mathbf{C}$ be an open set. Assume that $f\colon G \to \mathbf{C}$ is continuous, and that*

$$\int_{\partial\Delta} f = 0$$

for every solid triangle Δ entirely contained in G. Then $f \in \mathcal{H}(G)$.

Proof. Let f satisfy the assumptions in the theorem. The property of being holomorphic is a local property, so it suffices to prove that f is holomorphic in any ball $B(a,r) \subseteq G$. Take such a ball. Then the assumption implies $\int_{\partial\Delta} f = 0$ for any $\Delta \subseteq B(a,r)$. Thus we can repeat the argument in the proof of Theorem 4.17 to conclude that f has a primitive in $B(a,r)$. But then by Corollary 6.2 f is holomorphic in this ball. □

Cauchy's integral formula can be generalized as follows:

Theorem 6.4. *Let $G \subseteq \mathbf{C}$ be an open subset, and let $f \in \mathcal{H}(G)$. For $\overline{B}(a,r) \subseteq G$ and $z_0 \in B(a,r)$ we have Cauchy's integral formula for the n'th derivative*

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\partial B(a,r)} \frac{f(z)}{(z-z_0)^{n+1}}dz, \qquad n = 0,1,2,\ldots. \tag{6.3}$$

Proof. Let $\overline{B}(a,r) \subseteq G$ and $z_0 \in B(a,r)$. Theorem 6.1 implies that we have a Taylor expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!}(z-z_0)^k \tag{6.4}$$

valid in some ball $B(z_0, \rho)$. Choose r', $0 < r' < \rho$, such that $\overline{B}(z_0, r') \subset B(a, r)$. The series (6.4) is uniformly convergent on $\partial B(z_0, r')$. Thus we can interchange summation and integration in the following computation.

$$\int_{\partial B(z_0,r')} \frac{f(z)}{(z-z_0)^{n+1}} dz = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} \int_{\partial B(z_0,r')} (z-z_0)^{k-n-1} dz = \frac{f^{(n)}(z_0)}{n!} 2\pi i.$$

In the last step we used a computation similar to the one in Example 4.18. Using Example 5.6 we can change the integration contour from $\partial B(z_0, r')$ to $\partial B(a, r)$. Thus

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial B(z_0,r')} \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{n!}{2\pi i} \int_{\partial B(a,r)} \frac{f(z)}{(z-z_0)^{n+1}} dz,$$

which proves the result. □

It follows from the above results that a function $f \in \mathcal{H}(\mathbf{C})$ can be expanded in a convergent power series around any $z \in \mathbf{C}$ and that the radius of convergence is infinite. This class of functions is important enough to have a name.

Definition 6.5. A function $f \in \mathcal{H}(\mathbf{C})$ is called an entire function.

There are many important results on entire functions. We state one of them without proof.

Theorem 6.6 (Picard)**.** *Let f be a nonconstant entire function. Then either $f(\mathbf{C}) = \mathbf{C}$ or $f(\mathbf{C}) = \mathbf{C} \setminus \{a\}$ for some $a \in \mathbf{C}$. If f is not a polynomial, then $f^{-1}(\{w\})$ is an infinite set for all $w \in \mathbf{C}$, except for at most one w.*

Picard's theorem has as a consequence that a bounded entire function is a constant function. This result can be proved directly, using Theorem 6.4.

Theorem 6.7 (Liouville)**.** *A bounded entire function is a constant.*

Proof. Let $f \in \mathcal{H}(\mathbf{C})$ and assume that $|f(z)| \leq M$ for all $z \in \mathbf{C}$. For any $r > 0$ we can use (6.3) with $z_0 = 0$ to get

$$|f^{(n)}(0)| = \left| \frac{n!}{2\pi i} \int_{\partial B(0,r)} \frac{f(z)}{z^{n+1}} dz \right| \leq \frac{n!}{2\pi} \frac{M}{r^{n+1}} 2\pi r = \frac{M \cdot n!}{r^n}.$$

Letting $r \to \infty$ we find that $f^{(n)}(0) = 0$ for all $n \geq 1$. The result now follows from the Taylor expansion around zero. □

Liouville's theorem is a deep result. For instance it leads to the following proof of the fundamental theorem of algebra.

Theorem 6.8 (Fundamental theorem of algebra)**.** *Any polynomial*

$$p(z) = \sum_{k=0}^{n} a_k z^k$$

of degree $n \geq 1$ has at least one root in $\mathbf{C}$.

Proof. Let $p(z)$ be a polynomial of degree $n \geq 1$. Assume that $p(z) \neq 0$ for all $z \in \mathbf{C}$. Then

$$\frac{p(z)}{a_n z^n} = 1 + \sum_{k=0}^{n-1} \frac{a_k}{a_n} z^{k-n} \to 1 \quad \text{for} \quad |z| \to \infty,$$

hence we can find $r > 0$ such that

$$\left| \frac{p(z)}{a_n z^n} \right| \geq \frac{1}{2} \quad \text{for} \quad |z| \geq r.$$

This estimate implies

$$\frac{1}{|p(z)|} \leq \frac{2}{|a_n| r^n} \quad \text{for} \quad |z| \geq r.$$

Since $|p(z)|^{-1}$ is a continuous function, it is also bounded on the compact set $\overline{B}(0, r)$, and we have shown that p^{-1} is a bounded entire function. By Liouville's theorem $p(z)$ is a constant which contradicts the assumption that the degree is at least one. □

6.1 Exercises

1. Evaluate
$$\int_{\partial B(i,2)} \frac{e^z}{(z-1)^n} dz \quad \text{for all } n \geq 1.$$

2. Assume that f is an entire function and also satisfies $f' = af$ for some $a \in \mathbf{C}$. Prove that there exists $c \in \mathbf{C}$ such that
$$f(z) = c \exp(az), \quad z \in \mathbf{C}.$$

3. Let $f \in \mathcal{H}(G)$, and assume $f'(a) \neq 0$. Prove that there exists $r > 0$ such that f is injective on $B(a, r)$.

4. Let $G \subsetneq \mathbf{C}$ be an open subset, and let $a \in G$. Prove that $\inf\{|z - a| \,|\, z \in \mathbf{C} \setminus G\}$ is attained at some point, i.e. that the infimum is actually a minimum. *Hint:* Use the triangle inequality.

5. Use Theorem 6.3 to prove the following important result. Assume that $f_n \in \mathcal{H}(G)$ and $f_n \to f$ as $n \to \infty$, uniformly on all compact subsets of G. Then $f \in \mathcal{H}(G)$.

7 Meromorphic functions

We now study zeroes and singularities of holomorphic functions. We start by defining the order of a zero.

Theorem 7.1. *Let f be holomorphic in a domain G. Assume that $a \in G$ is a zero of f, i.e. $f(a) = 0$. Then either $f^{(n)}(a) = 0$ for all $n = 1, 2, \ldots$, and in this case $f(z) = 0$ for all $z \in G$, or there exists a smallest n, $n \geq 1$, such that $f^{(n)}(a) \neq 0$. In the latter case a is called a zero of the n'th order, and the function defined by*

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-a)^n}, & z \in G \setminus \{a\}, \\ \dfrac{f^{(n)}(a)}{n!}, & z = a, \end{cases}$$

is holomorphic in G, and satisfies the equation $f(z) = (z-a)^n g(z)$, $z \in G$. Furthermore, $g(a) \neq 0$.

Proof. We start by proving that $f^{(n)}(a) = 0$, $n = 0, 1, 2, \ldots$, for some $a \in G$ implies that the function f is identically zero in G. Let

$$A = \bigcap_{n=0}^{\infty} \{z \in G \,|\, f^{(n)}(z) = 0\}.$$

Since $f^{(n)}$ is continuous for all n, the set A is the intersection of closed subsets of G, hence A is a closed subset of G. If $z_0 \in A$, then the Taylor series around this point is the zero series, and it follows that f is identically zero in the largest ball $B(z_0, \rho) \subseteq G$. But then $B(z_0, \rho) \subseteq A$, and A is an open subset of G. Since G is assumed to be a domain, it is a connected set, and therefore either $A = \emptyset$ or $A = G$. Thus if A is nonempty, it equals G, and we have proved the first statement in the theorem.

We now assume that f is not identically zero in G. By the definition of n we have $f^{(k)}(a) = 0$, $k = 0, \ldots, n-1$. Let ρ be the radius of the largest open ball $B(a, \rho)$ contained in G. The Taylor expansion then has the form

$$f(z) = \sum_{k=n}^{\infty} \frac{f^{(k)}(a)}{k!}(z-a)^k = (z-a)^n \sum_{k=0}^{\infty} \frac{f^{(k+n)}(a)}{(k+n)!}(z-a)^k.$$

The function

$$g(z) = \begin{cases} \dfrac{f(z)}{(z-a)^n}, & z \in G \setminus \{a\}, \\ \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k+n)}(a)}{(k+n)!}(z-a)^k, & z \in B(a, \rho), \end{cases}$$

is a welldefined function, holomorphic in G. By definition it satisfies $f(z) = (z-a)^n g(z)$, $z \in G$, and $g(a) = (n!)^{-1} f^{(n)}(a) \neq 0$. □

The theorem shows that one can find the order of the zero a of $f \not\equiv 0$ as the largest n, for which one has a factorization $f(z) = (z-a)^n g(z)$ for some $g \in \mathcal{H}(G)$. We have the following result concerning the set of zeroes of a holomorphic function. Let $A \subseteq G$ be a subset of domain in $\mathbf{C}$. Recall that a point $a \in A$ is said to be an isolated point of A in G, if there exists a $\delta > 0$ such that $B(a, \delta) \subseteq G$ and $B(a, \delta) \cap A = \{a\}$.

Theorem 7.2. *Assume that f is holomorphic in a domain G. Let $Z(f)$ denote the set of zeroes of f in G. Then precisely one of the following three cases occurs.*

1. *$Z(f) = \emptyset$, which means that f has no zeroes in G.*

2. *$Z(f) = G$, which means that f is identically zero in G.*

3. *$Z(f)$ consists of at most a countable number of isolated points in G.*

Proof. If $f(a) = 0$ and $f \not\equiv 0$, then we can find $g \in \mathcal{H}(G)$ and an integer n, such that $f(z) = (z-a)^n g(z)$ and $g(a) \neq 0$. Since g is continuous, we can find a $\delta_a > 0$ such that $g(z) \neq 0$ for all $z \in B(a, \delta_a) \subseteq G$. Thus $Z(f) \cap B(a, \delta_a) = \{a\}$ and a has been shown to be an isolated point in G. To show that $Z(f)$ is at most countable, we use the Lindelöf covering theorem [1, Theorem 3.28]. The open covering $\{B(a, \delta_a)\}_{a \in Z(f)}$ of $Z(f)$ can then be replaced by an at most countable covering. It follows that $Z(f)$ is at most countable. □

The following theorem is often used to show that two holomorphic functions are identical in a certain domain.

Theorem 7.3 (Identity theorem). *Let G be a domain, and let $f, g \in \mathcal{H}(G)$. Assume that $A \subseteq G$ has an accumulation point in G. If $f(z) = g(z)$ for all $z \in A$, then $f(z) = g(z)$ for all $z \in G$.*

Proof. The set of zeroes $Z(f - g)$ is a closed set. By assumption $A \subseteq Z(f - g)$, and therefore the accumulation point a of A belongs to $Z(f - g)$. But then a is a zero, which is not isolated. By Theorem 7.2 we must have $Z(f - g) = G$, which proves the theorem. □

This theorem is often applied to the case where A is a subset of the real axis. We have the following example.

Example 7.4. All usual trigonometric identities hold also in the complex domain, if the expressions entering into the identities are holomorphic functions. As an example, we have $\sin^2(z) + \cos^2(z) = 1$ for all $z \in \mathbf{C}$. Both the left hand side and the right hand side are holomorphic in $\mathbf{C}$, and the formula is known to be valid for all real z.

We now study the singularities of holomorphic functions.

Definition 7.5. Let $G \subseteq \mathbf{C}$ be an open subset and let $a \in G$. If $f \in \mathcal{H}(G \setminus \{a\})$ then a is said to be an isolated singularity of f. If a value can be assigned to f in this point such that f becomes holomorphic in G, then a is said to be a removable singularity.

If $f \in \mathcal{H}(G)$ and $f \not\equiv 0$, then the function $1/f$ is holomorphic in the open set $G \setminus Z(f)$, and all $a \in Z(f)$ are isolated singularities of $1/f$.

Given a holomorphic function with certain isolated singularities, the removable singularities are removed by assigning values at these points. As an example take the function $f(z) = z^{-1}\sin(z)$. It can be extended from $\mathbf{C} \setminus \{0\}$ to all of $\mathbf{C}$ by defining $f(0) = \lim_{z \to 0} z^{-1}\sin(z) = 1$. We have the following result, which can be used to decide whether a given singularity is removable.

Theorem 7.6. *Let $G \subseteq \mathbf{C}$ be an open subset, and let $a \in G$. If $f \in \mathcal{H}(G \setminus \{a\})$ is bounded in $B(a, r) \setminus \{a\} \subseteq G \setminus \{a\}$ for some $r > 0$, then the singularity of f at a is removable.*

Proof. Define a function $h: G \to \mathbf{C}$ by $h(a) = 0$ and $h(z) = (z - a)^2 f(z)$ for $z \in G \setminus \{a\}$. Then h is holomorphic in $G \setminus \{a\}$, and for $z \neq a$ we have

$$\frac{h(z) - h(a)}{z - a} = (z - a) f(z),$$

which has the limit 0 for $z \to a$, since f is assumed to be bounded in $B(a, r) \setminus \{a\}$ for some $r > 0$. Thus we have shown that h is differentiable in the complex sense at a with $h'(a) = 0$.

We now apply Theorem 7.1 to the function $h \in \mathcal{H}(G)$. If $f \equiv 0$ on $G \setminus \{a\}$, the singularity is removed by setting $f(a) = 0$. If $f \not\equiv 0$ on $G \setminus \{a\}$ then h has a zero of order at least 2 at a, and we can find a function $g \in \mathcal{H}(G)$, such that $h(z) = (z - a)^2 g(z)$ for all $z \in G$. Thus g is a holomorphic extension of f to G, and we have proved that the singularity at a is removable. □

If a is a singularity which is not removable, then Theorem 7.6 shows that $f(B(a, r) \setminus \{a\})$ is an unbounded set for any sufficiently small $r > 0$. As a consequence, the limit $\lim_{z \to a} f(z)$ does not exist. One could then try to investigate whether the function $(z - a)^m f(z)$ has a removable singularity at a, if m is a sufficiently large integer. In such a case a is said to be a pole of f.

Definition 7.7. An isolated singularity a of a holomorphic function $f \in \mathcal{H}(G \setminus \{a\})$ is said to be a pole of order $m \in \mathbf{N}$, if $(z - a)^m f(z)$ has a limit different from zero as $z \to a$. A pole of order 1 is called a simple pole.

We should note that the order of a pole is determined uniquely. If $\lim_{z\to a}(z-a)^m f(z) = c \neq 0$, then $\lim_{z\to a}(z-a)^k f(z) = 0$ for $k > m$, and for $k < m$ the limit cannot exist.

Assume that $f \in \mathcal{H}(G \setminus \{a\})$ has a pole of order m in a. The function defined by

$$g(z) = \begin{cases} (z-a)^m f(z), & z \in G \setminus \{a\}, \\ \lim_{z\to a}(z-a)^m f(z), & z = a, \end{cases}$$

will then be a function holomorphic in G. Such a function has a power series expansion around a, which we write as $\sum_{n=0}^{\infty} a_n (z-a)^n$. The series is convergent in the largest open ball $B(a,\rho)$ contained in G. As a consequence we have

$$f(z) = \frac{a_0}{(z-a)^m} + \frac{a_1}{(z-a)^{m-1}} + \cdots + \frac{a_{m-1}}{z-a} + \sum_{k=0}^{\infty} a_{m+k}(z-a)^k$$

for all $z \in B(a,\rho) \setminus \{a\}$. We define $p(z) = \sum_{k=1}^{m} a_{m-k} z^k$ and call $p((z-a)^{-1})$ the principal part of f at a. Then f minus its principal part has a removable singularity at a. Thus the singularity is localized in the principal part of f at a.

A singularity, which is neither removable nor a pole, is called an essential singularity. In the neighborhood of an essential singularity the behavior of f is very complicated. Two examples are $\sin(1/z)$ and $\exp(1/z)$, which both are holomorphic in $\mathbf{C} \setminus \{0\}$, and both have an essential singularity at 0.

We will study holomorphic functions with singularities, but we want to avoid the complicated essential singularities.

Definition 7.8. Let $G \subseteq \mathbf{C}$ be a domain, and let $f\colon G \setminus P \to \mathbf{C}$ be a holomorphic function with isolated singularities in P. If all points in P are poles, then f is said to be meromorphic in G.

Meromorphic functions are often given as the quotient of two holomorphic functions. Assume $f, g \in \mathcal{H}(G)$ and $g \not\equiv 0$. If $f \equiv 0$ then by convention the quotient is the zero function. We will therefore assume $f \not\equiv 0$. Since $Z(g)$ is an isolated set in G, the function $h = f/g$ is holomorphic in $G \setminus Z(g)$ with isolated singularities in $Z(g)$. Let $a \in Z(g)$. Then from Theorem 7.1 we find that we can write

$$f(z) = (z-a)^p f_1(z), \quad g(z) = (z-a)^q g_1(z)$$

with $f_1, g_1 \in \mathcal{H}(G)$, $f_1(a) \neq 0$, $g_1(a) \neq 0$, and $q \geq 1$ is the order of the zero of g at a. Furthermore, $p \geq 0$ is zero, if $f(a) \neq 0$, and otherwise equal to the order of the zero of f at a. Choose $r > 0$ sufficiently small, such that $g_1(z) \neq 0$ for all $z \in B(a,r) \subseteq G$. Then

$$h(z) = (z-a)^{p-q} \frac{f_1(z)}{g_1(z)}$$

for all $z \in B(a,r) \setminus \{a\}$. Thus we have shown that h is meromorphic in G. The poles are those a in $Z(g)$, where either $f(a) \neq 0$, or the order of the zero of f at a is strictly smaller than the order of the zero of g.

One can prove that any function meromorphic in G can be expressed as the quotient of two holomorphic functions in G. A special class of meromorphic functions are the rational functions, which are the functions that can be expressed as the quotient of two polynomials.

The sum and the product of two meromorphic functions is again meromorphic, since the union of the two pole sets is again a set of isolated points in G. Obviously, some of the poles in the sum or product may actually be removable singularities.

In the terminology used in algebra the meromorphic functions in a fixed domain G constitute a commutative field.

7.1 Exercises

1. Let G be a domain in $\mathbf{C}$. Assume that $f \in \mathcal{H}(G)$ only has a finite number of zeroes in G. Prove that there exist a polynomial $p(z)$, and a function $\varphi \in \mathcal{H}(G)$ without zeroes, such that $f(z) = p(z)\varphi(z)$ for $z \in G$.

2. Determine $a \in \mathbf{C}$ such that the function $\sin(z) - z(1 + az^2)\cos(z)$ has a zero of order 5 at $z = 0$.

8 The residue theorem

We start by defining the residue at a pole of a meromorphic function.

Definition 8.1. Let $f\colon G \setminus P \to \mathbf{C}$ be a function meromorphic in G with poles in P. Let $a \in P$ be a pole of order m. The coefficient to the term $(z-a)^{-1}$ in the principal part of f at a is called the residue and is denoted by $\operatorname{Res}(f, a)$.

We have at a pole a a representation

$$f(z) = \frac{c_m}{(z-a)^m} + \cdots + \frac{c_1}{z-a} + \varphi(z),$$

where φ is a function holomorphic in a small ball around a, and furthermore φ is meromorphic in G with poles in $P \setminus \{a\}$.

The definition of a pole of order m implies $c_m = \lim_{z\to a}(z-a)^m f(z) \neq 0$. As a consequence, if a is a simple pole, then $\operatorname{Res}(f, a) \neq 0$.

Theorem 8.2. *Let $f\colon G \setminus P \to \mathbf{C}$ be meromorphic in a domain G with poles in P. Let $a_0 \in P$. Then we have*

$$\operatorname{Res}(f, a_0) = \frac{1}{2\pi i}\int_{\partial B(a_0,r)} f(z)dz,$$

where $r > 0$ is chosen such that $\overline{B}(a_0, r) \subseteq G \setminus (P \setminus \{a_0\})$.

Proof. Assume that a_0 is a pole of order m. Let $B(a_0, \rho)$ denote the largest open ball contained in $G \setminus (P \setminus \{a_0\})$, We use

$$f(z) = \frac{c_m}{(z-a_0)^m} + \cdots + \frac{c_1}{z-a_0} + \varphi(z),$$

as above. Since φ is holomorphic in the starshaped domain $B(a_0, \rho)$, it has a primitive in $B(a_0, \rho)$. Let $r > 0$ be chosen as in the theorem. Then $\int_{\partial B(a_0,r)} \varphi(z)dz = 0$. Analogously we have $\int_{\partial B(a_0,r)} (z-a_0)^{-k}dz = 0$ for $k \geq 2$, since $(z-a_0)^{-k}$ for $k \geq 2$ has a primitive in $\mathbf{C} \setminus \{a_0\}$. We now have

$$\int_{\partial B(a_0,r)} f(z)dz = c_1 \int_{\partial B(a_0,r)} \frac{1}{z-a_0}dz = 2\pi i c_1.$$

Since $c_1 = \operatorname{Res}(f, a_0)$, the result is proved. □

In Section 5 we have seen that the circuit $\partial B(a, r)$ in some cases can be replaced by another circuit enclosing the singularity. We have the following result.

Proposition 8.3. *Let $\gamma\colon [a, b] \to \mathbf{C}$ be a simple, closed, positively oriented circuit. For $z_0 \notin \gamma^* = \gamma([a, b])$ we have*

$$\int_\gamma \frac{1}{z-z_0}dz = \begin{cases} 2\pi i, & \text{if } \gamma \text{ surrounds } z_0,\\ 0, & \text{otherwise.}\end{cases}$$

Proof. The proof is only given in a special case. We assume that we can define piecewise continuous functions $r\colon [a,b] \to (0,\infty)$ and $\varphi\colon [a,b] \to \mathbf{R}$ such that the given circuit can be represented as

$$\gamma(t) = z_0 + r(t)e^{i\varphi(t)}, \quad t \in [a,b].$$

In the case where $r(t)$ and $\varphi(t)$ are differentiable in $[a,b]$ we have

$$\begin{aligned}\int_\gamma \frac{1}{z-z_0}dz &= \int_a^b \frac{\gamma'(t)}{\gamma(t)-z_0}dt \\ &= \int_a^b \frac{r'(t)e^{i\varphi(t)} + r(t)i\varphi'(t)e^{i\varphi(t)}}{r(t)e^{i\varphi(t)}}dt \\ &= \int_a^b \frac{d}{dt}[\log(r(t))]dt + i\int_a^b \varphi'(t)dt \\ &= i[\varphi(b) - \varphi(a)].\end{aligned}$$

In the general case we have to split the integral into a sum of integrals over the subintervals of $[a,b]$, where $r(t)$ and $\varphi(t)$ both are differentiable. The result now follows from the fact that $\varphi(b)-\varphi(a) = 2\pi$, if the circuit circumscribes z_0, and zero otherwise. This property is easily verified for explicit circuits, such as circles and polygonal paths. The general case is proved by using a deep result called the Jordan curve theorem. □

We can now explain the term ‘residue’. It is the remainder (up to a factor $2\pi i$) left in the contour integral, when integrating a meromorphic function along a circuit circumscribing the singularity once. In French remainder is ‘residu’.

We need the following result in the proof of the main theorem in this section.

Lemma 8.4. *Let $\gamma\colon [a,b] \to G$ be a simple closed circuit in a starshaped domain $G \subseteq \mathbf{C}$. Then there exists a bounded starshaped domain G_1 such that $\gamma^* \subseteq G_1$ and $\overline{G}_1 \subseteq G$.*

Proof. To each $z \in \gamma^*$ there exists $r_z > 0$ such that $B(z, r_z) \subseteq G$. We introduce the covering $\{B(z, r_z/2)\}_{z\in\gamma^*}$ of γ^*. Since γ^* is a compact set, this covering can be replaced by a finite covering $\gamma^* \subseteq \cup_{k=1}^n B(z_k, r_k/2) = A \subseteq G$. Assume that G is starshaped around $a \in G$. We can find $r > 0$ such that $B(a,r) \subseteq G$. We now define

$$G_1 = \bigcup_{z\in A} L(a,z) \bigcup B(a, \frac{r}{2}).$$

It follows from this definition that G_1 is starshaped around a and that $\gamma^* \subseteq A \subseteq G_1$. Furthermore, G_1 is contained in the ball with center a and radius equal to $R = \max_{k=1\dots n}\{|z_k - a| + r_k/2\} + r/2$, implying that G_1 is bounded. Finally we note that

$$\overline{G}_1 \subseteq \bigcup_{z\in\cup_{k=1}^n B(z_k, r_k)} L(a,z) \bigcup B(a,r) \subseteq G.$$

□

We can now prove the main theorem in this section.

Theorem 8.5 (Cauchy’s residue theorem)**.** *Let f be meromorphic in a starshaped domain G. Let P denote the poles of f. Let γ be a simple, closed, positively oriented circuit in $G \setminus P$. Let $a_1, \dots, a_n$, denote the poles surrounded by γ. Then*

$$\int_\gamma f(z)dz = 2\pi i \sum_{k=1}^n \mathrm{Res}(f, a_k).$$

Proof. We use Lemma 8.4 to determine a bounded and starshaped domain G_1, such that $\gamma^* \subseteq G_1$ and $\overline{G}_1 \subseteq G$. The set of poles P is a closed subset of G with no accumulation points in G. Thus $P \cap \overline{G}_1$ has no accumulation points in G. It now follows from the Bolzano-Weierstrass theorem [3] that the set $P_1 = P \cap G_1$ is a finite set. We write

$$P_1 = \{a_1, \dots, a_n, a_{n+1}, \dots, a_{n+m}\},$$

where $a_{n+1}, \dots, a_{n+m}$ are those poles in G_1 not surrounded by γ.

Let p_k denote the principal part of f at a_k, $k = 1, \dots, n+m$. Let $\varphi = f - \sum_{k=1}^{m+n} p_k$. This function has a removable singularity at each of the points $a_1, \dots, a_{n+m}$. By assigning the right values in these points we get a function $\varphi \in \mathcal{H}(G_1)$. Since G_1 is starshaped, Corollary 5.5 shows that φ has a primitive in G_1. Thus $\int_\gamma \varphi(z)dz = 0$. Using the definitions we have shown

$$\int_\gamma f(z)dz = \sum_{k=1}^{n+m} \int_\gamma p_k(z)dz.$$

The principal part p_k is holomorphic in $\mathbf{C} \setminus \{a_k\}$, and has a representation

$$p_k(z) = \frac{c^k_{\tilde{m}}}{(z-a_k)^{\tilde{m}}} + \cdots + \frac{c^k_2}{(z-a_k)^2} + \frac{\mathrm{Res}(h, a_k)}{z-a_k}.$$

For each k we have $\int_\gamma (z-a_k)^{-j}dz = 0$ for $j \geq 2$, since $(z-a_k)^{-j}$ for $j \geq 2$ has a primitive in $\mathbf{C} \setminus \{a_k\}$, by Proposition 8.3. We also have that $\int_\gamma (z-a_k)^{-1}dz$ equals $2\pi i$, if γ surrounds a_k, and 0 otherwise. □

We will now give some prescriptions for finding the residue of a meromorphic function, which is represented as $h = f/g$. We assume that h has a pole in a and want to calculate $\mathrm{Res}(h, a)$:

1. Assume that h has a simple pole at a. Then $\mathrm{Res}(h, a) = \lim_{z \to a}(z-a)h(z)$.
2. Assume $f(a) \neq 0$, $g(a) = 0$, $g'(a) \neq 0$. Then $\mathrm{Res}(h, a) = f(a)/g'(a)$. This result follows from the first result, since h has a simple pole in a, and

$$\mathrm{Res}(h, a) = \lim_{z \to a} f(z)\frac{z-a}{g(z)} = \lim_{z \to a} f(z)\left(\frac{g(z)-g(a)}{z-a}\right)^{-1} = \frac{f(a)}{g'(a)}.$$

3. Assume that h has a pole of order m at a. Then

$$\mathrm{Res}(h, a) = \frac{H^{(m-1)}(a)}{(m-1)!},$$

where $H(z) = (z-a)^m h(z)$, such that H has a removable singularity at a. This result is proved by using the representation

$$h(z) = \frac{c_m}{(z-a)^m} + \cdots + \frac{c_1}{z-a} + \varphi(z)$$

of h in a ball around a.

Example 8.6. The function

$$h(z) = \frac{z\sin(z)}{1-\cos(z)}$$

is meromorphic in $\mathbf{C}$. The denominator has zeroes at $2j\pi$, $j \in \mathbf{Z}$. All zeroes are of order 2. The numerator has a zero of order 2 at $z = 0$ and zeroes of order 1 at $z = j\pi$, $j \in \mathbf{Z} \setminus \{0\}$. It follows that $z = 0$ is a removable singularity and that $z = 2j\pi$, $j \in \mathbf{Z} \setminus \{0\}$, are simple poles.

The value to be assigned at the removable singularity is found using the power series expansions for sin and cos:

$$\lim_{z \to 0} h(z) = \lim_{z \to 0} \frac{z(z - \frac{z^3}{3!} + \cdots)}{\frac{z^2}{2!} - \frac{z^4}{4!} + \cdots} = 2.$$

The residue at a pole $2j\pi$, $j \neq 0$, is found using $w = z - 2j\pi$ and the periodicity of trigonometric functions:

$$\mathrm{Res}(h, 2j\pi) = \lim_{z \to 2j\pi} (z - 2j\pi) h(z) = \lim_{w \to 0} \frac{w(w + 2j\pi)\sin(w)}{1 - \cos(w)} = 4j\pi.$$

8.1 Exercises

1. Carry out the details in the proofs of the three prescriptions for determining residues given above.
2. Find the poles and their orders, and calculate the residues, for each of the following functions:
   (a) $f(z) = \dfrac{1}{\sin(z)}$,
   (b) $g(z) = \dfrac{z^2}{(z^2+1)^2}$,
   (c) $h(z) = \dfrac{1}{e^{z^2} - 1}$.
3. Let $f, g \in \mathcal{H}(G)$. Assume that f has a zero of order $n > 0$ at $a \in G$, and that g has a zero of order $n + 1$ at a. Prove that f/g has a simple pole at a, and show that
$$\mathrm{Res}(f/g, a) = (n+1)\frac{f^{(n)}(a)}{g^{(n+1)}(a)}.$$
4. Prove that the function $\sin(z^{-1})$ has an essential singularity at $z = 0$.

9 Applications of the residue theorem

We have the following result for meromorphic functions.

Theorem 9.1. *Let h be a meromorphic function defined on a starshaped domain G. Let γ be a simple, closed, positively oriented circuit in G, which does not intersect any of the poles or zeroes of h in G. Let $N_\gamma(P)$ denote the sum of the orders of the poles of h surrounded by γ. Let $N_\gamma(Z)$ denote the sum of the orders of the zeroes of h surrounded by γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{h'(z)}{h(z)} dz = N_\gamma(Z) - N_\gamma(P).$$

Proof. Let D denote all zeroes and poles of h in G. The function h'/h is holomorphic in $G \setminus D$. We now prove that h'/h is meromorphic in G with poles contained in D. Let P denote the poles of h in G. Assume that a is a zero of order n of h. Then we have the expansion

$$h(z) = a_n(z-a)^n + a_{n+1}(z-a)^{n+1} + \cdots, \quad a_n \neq 0,$$

valid in the largest ball $B(a,\rho)$ contained in $G \setminus P$. Thus

$$h'(z) = na_n(z-a)^{n-1} + (n+1)a_{n+1}(z-a)^n + \cdots .$$

It follows that in this case the function h'/h has a simple pole at a with residue equal to n. Assume now that a is a pole of order m of h. Then we have the expansion

$$h(z) = a_{-m}(z-a)^{-m} + a_{-m+1}(z-a)^{-m+1} + \cdots , \quad a_{-m} \neq 0$$

valid in $B(a,\rho)\setminus\{a\}$, where $B(a,\rho)$ is the largest open ball contained in $G \setminus (P \setminus \{a\})$. We now have

$$h'(z) = -ma_{-m}(z-a)^{-m-1} - (m-1)a_{-m+1}(z-a)^{-m} - \cdots .$$

We conclude that the function h'/h in this case has a simple pole at a, with residue equal to $-m$. The result in the theorem now follows from the residue theorem. □

We now show how to use the residue theorem to evaluate certain types of definite integrals. We start by defining improper Riemann integrals, which are Riemann integrals over finite or infinite open intervals.

For $a, b \in \mathbf{R} \cup \{\pm\infty\}$, $a < b$, assume that f is Riemann integrable over all finite closed subintervals of (a,b). Then we define

$$\int_{(a,b)} f(x)dx = \lim_{c\to a+, d\to b-} \int_{[c,d]} f(x)dx$$

if the limit exists.

One of the ideas used in computing an improper Riemann integral $\int_{-\infty}^{\infty} f(x)dx$ can be described briefly as follows: Let γ be a closed circuit in $\mathbf{C}$, which contains the interval $[-R,R]$, for example this line segment concatenated with the semi-circle in the upper half plane connecting R with $-R$. Suppose we can find a meromorphic function F which agrees with f on the real axis. The integral from $-R$ to R plus the integral along the semi-circle then equals $2\pi i$ times the residues at some of the poles of F in the upper half plane. One then tries to evaluate the limit $R \to \infty$. In many cases the integral along the semi-circle will tend to zero. In the limit one then gets the value of the integral over the real axis. Let us illustrate this procedure in the next proposition.

Proposition 9.2. *Let f be a rational function*

$$f(z) = \frac{p(z)}{q(z)} = \frac{a_0 + a_1 z + \cdots + a_m z^m}{b_0 + b_1 z + \cdots + b_n z^n}, \quad a_m \neq 0, \quad b_n \neq 0.$$

Assume that $n \geq m+2$ and that f has no poles on the real axis. Then the following improper integral exists and is evaluated as shown.

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, z_j) = -2\pi i \sum_{j=1}^{l} \mathrm{Res}(f, w_j),$$

where $z_1, \ldots, z_k$ are the poles of f located in the upper half plane and $w_1, \ldots, w_l$ the poles in the lower half plane.

Proof. We start by noting that

$$\lim_{|z|\to\infty} z^{n-m} f(z) = a_m/b_n.$$

We can then determine $R_0 > 0$ such that

$$|z|^{n-m}|f(z)| \leq M = |a_m/b_n| + 1 \quad \text{for} \quad |z| \geq R_0.$$

In particular, we have

$$|f(z)| \leq \frac{M}{|z|^2} \quad \text{for} \quad |z| \geq \max\{1, R_0\}.$$

Choose $R > 0$ sufficiently large such that it is larger than $\max\{1, R_0\}$ and such that all the poles of f are contained in $B(0, R)$, then the residue theorem gives (with γ_1 the line segment and the semi-circle in the upper half plane traversed in the positive direction)

$$\int_{\gamma_1} f(z)dz = \int_{-R}^{R} f(x)dx + \int_0^{\pi} f(Re^{i\theta})iRe^{i\theta}d\theta = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f, z_j).$$

Analogously, with γ_2 denoting the line segment traversed from R to $-R$, and the semi-circle in the lower half plane, traversed in the positive direction, we have

$$\int_{\gamma_2} f(z)dz = \int_{R}^{-R} f(x)dx + \int_{\pi}^{2\pi} f(Re^{i\theta})iRe^{i\theta}d\theta = 2\pi i \sum_{j=1}^{l} \operatorname{Res}(f, w_j).$$

Since

$$\left| \int_0^{\pi} f(Re^{i\theta})iRe^{i\theta}d\theta \right| \leq \pi \frac{RM}{R^2},$$

the contribution from the semi-circle tends to zero for $R \to \infty$. A similar result holds for the integral over the semi-circle in the lower half plane. □

Example 9.3. We consider the rational function $f(z) = z^2(z^2+1)^{-2}$. It has no poles on the real axis, and in the upper half plane it has a pole at $z = i$ with residue equal to $-i/4$. Proposition 9.2 then yields

$$\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = 2\pi i \operatorname{Res}(f, i) = \frac{\pi}{2}.$$

The method in Proposition 9.2 can be applied to many other classes of functions. We can state the following result.

Proposition 9.4. *Let f be meromorphic in $\mathbf{C}$ with no poles on the real axis, and with at most a finite number of poles in the upper half plane, denoted by $z_1, \ldots, z_k$. If*

$$\max_{0 \leq t \leq \pi} |f(Re^{it})| \to 0 \quad \textit{for } R \to \infty,$$

then the improper Riemann integral $\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx$ exists for any $\lambda > 0$, and is given by

$$\int_{-\infty}^{\infty} f(x)e^{i\lambda x}dx = 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f(z)e^{i\lambda z}, z_j).$$

Proof. Let γ denote the closed circuit consisting of the line segment from $-R$ to R and the semi-circle $|z| = R$, $\operatorname{Im}(z) \geq 0$, in the upper half plane. Assume that R is sufficiently large, such that this circuit surrounds all the poles in the upper half plane. Using the residue theorem we find

$$\begin{aligned}\int_{\gamma} f(z)e^{i\lambda z}dz &= \int_{-R}^{R} f(x)e^{i\lambda x}dx + \int_0^{\pi} f(Re^{it})e^{i\lambda Re^{it}}iRe^{it}dt \\ &= 2\pi i \sum_{j=1}^{k} \operatorname{Res}(f(z)e^{i\lambda z}, z_j).\end{aligned}$$

We have

$$I_R = \left| \int_0^\pi f(Re^{it}) e^{i\lambda Re^{it}} iRe^{it} dt \right| \leq \max_{0 \leq t \leq \pi} |f(Re^{it})| \int_0^\pi Re^{-\lambda R \sin(t)} dt.$$

Since $\sin(t) \geq 2t/\pi$ for $t \in [0, \pi/2]$, we get with $a = 2\lambda R/\pi$

$$\begin{aligned}
\int_0^\pi e^{-\lambda R \sin(t)} dt &= 2 \int_0^{\pi/2} e^{-\lambda R \sin(t)} dt \\
&\leq 2 \int_0^{\pi/2} e^{-at} dt = \frac{2}{a}(1 - e^{-a\frac{\pi}{2}}) \\
&\leq \frac{2}{a} = \frac{\pi}{\lambda R},
\end{aligned}$$

which implies the estimate

$$I_R \leq \frac{\pi}{\lambda} \max_{0 \leq t \leq \pi} |f(Re^{it})|.$$

By assumption the right hand side tends to zero for $R \to \infty$. □

Example 9.5. The rational function $f(z) = z(z^2+1)^{-1}$ has no poles on the real axis. In the upper half plane it has a simple pole at $z = i$. We have the estimate

$$\max_{0 \leq t \leq \pi} |f(Re^{it})| \leq \frac{R}{R^2 - 1} \quad \text{for } R > 1.$$

Thus all assumptions in Proposition 9.4 are satisfied, and we get

$$\int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx = 2\pi i \operatorname{Res}(f(z)e^{i\lambda z}, i) = \pi i e^{-\lambda} \quad \text{for } \lambda > 0.$$

Taking the imaginary part on both sides we get

$$\int_{-\infty}^{\infty} \frac{x \sin(\lambda x)}{x^2 + 1} dx = \pi e^{-\lambda} \quad \text{for } \lambda > 0.$$

Finally we note that an integral of the form

$$\int_0^{2\pi} f(\cos(t), \sin(t)) dt$$

can be rewritten as a contour integral over the unit circle

$$\int_{\partial B(0,1)} f\left(\frac{1}{2}(z + \frac{1}{z}), \frac{1}{2i}(z - \frac{1}{z})\right) \frac{1}{iz} dz,$$

since $z = e^{it}$, $t \in [0, 2\pi]$, is a parametrization of the unit circle $\partial B(0,1)$. In some cases one can then use the residue theorem to find the contour integral.

Example 9.6. For $a > 1$ we consider the integral

$$\int_0^{2\pi} \frac{dt}{a + \cos(t)} = \int_{\partial B(0,1)} \frac{dz}{iz(a + \frac{1}{2}(z + \frac{1}{z}))} = -2i \int_{\partial B(0,1)} \frac{dz}{z^2 + 2az + 1}.$$

The denominator in the last integrand can be factored as $(z - q)(z - p)$, where $q < -1 < p < 0$ are the numbers $-a \pm \sqrt{a^2 - 1}$. Thus the integrand has a simple pole at $z = p$ inside the unit circle, and the residue is given by $(p - q)^{-1}$, which implies

$$\int_0^{2\pi} \frac{dt}{a + \cos(t)} = (-2i)(2\pi i) \frac{1}{p - q} = \frac{2\pi}{\sqrt{a^2 - 1}}.$$

9.1 Exercises

1. Show that $\int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$.

2. Show that $\int_{0}^{\infty} \frac{x\,dx}{1+x^4} = \frac{\pi}{4}$.

3. Show that $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)(x^2+4)} dx = \frac{\pi}{3}$.

4. Show that $\int_{-\infty}^{\infty} \frac{e^{i\pi x}}{x^2-2x+2} dx = -\pi e^{-\pi}$.

5. Show that for any $a > 1$ we have $\int_{0}^{2\pi} \frac{\cos(x)}{a+\cos(x)} dx = 2\pi(1 - \frac{a}{\sqrt{a^2-1}})$.

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